Learning objectives: understand internal waves, explain storm surge, and identify the effect of the Earth’s rotation

1. Internal waves; concepts - barotropic and baroclinic modes;
2. Storm surges
3. Effects of rotation, Rossby radius of deformation
Previous classes: we discussed surface gravity waves. What assumptions did we make to isolate “surface” gravity waves?

**Constant density – homogeneous ocean**

**Observed ocean density:**

\[
\text{Sigma}(0) = \text{(density} - 1000) \text{ kg/m}^3
\]

Potential density relative to surface
Concepts: Vertical profiles of density, T, S

Thermocline, halocline and pycnocline
1. Internal gravity waves in density stratified ocean

Largest amplitudes:
in pycnocline
Internal gravity waves in density stratified ocean

a) A 2-layer model

\[ \rho_1 \quad H_1 \quad 100-200\text{m (mixed layer)} \quad \text{Pycnocline} \]

\[ \rho_2 \quad H_2 \quad 3800\text{m (deep ocean below Pycnocline)} \]
Barotropic and baroclinic modes

The 2-layer system has 2 vertical normal modes. Total Solution is the superposition of the two modes.

**Barotropic mode:** independent of z, which represents vertically-averaged motion.

*Restoring force: gravity g*

**Baroclinic mode:** vertical shear flow, and vertically-integrated transport is zero. **Restoring force is reduced gravity**, 

\[ g' = \frac{g(\rho_2 - \rho_1)}{\rho_2} \sim 0.03 \text{m/s}^2 \]

Why? At ocean surface – air density is \(<<\) water density & thus ignored, and restoring force is gravity (g); in the pycnocline, the restoring force is reduced gravity \(g'\): due to the small difference between the water density above and below
Barotropic mode

\[ Z=0 \]

\[ \begin{align*}
C_0 &= \sqrt{g(H_1 + H_2)} = \sqrt{gD} \sim 200 \text{m/s} \\
\frac{h}{\eta} &= \frac{H_2}{H_1 + H_2} \leq 1 \sim 1, \quad \frac{u_2}{u_1} \sim 1
\end{align*} \]
Baroclinic mode

\[ C_1 = \sqrt{g' \frac{H_1 H_2}{H_1 + H_2}} \sim 2 - 3 \text{m/s} \]

\[ \frac{h}{\eta} = -\frac{g}{g'} \sim -300 \text{ (internal gravity waves)} \]
A $\frac{1}{2}$ - layer model

Deep Ocean: infinitely deep and $\nabla p = 0$

The system has only one baroclinic mode; No barotropic mode since we assumed $\nabla p = 0$ below the pycnocline. *We can also view it as the deep ocean is infinitely deep.*
2. Storm surge

Coastal shallow water: amplify. Why?
Basic dynamics:
(i) winds associated with storms or hurricanes pile up the water – **wind surge**;
(ii) low sea level pressure at the storm center (minimal comparing with wind) – **pressure surge**;
(iii) Shallow, gently sloping coastal region: intensify;
(iv) Overlapping with high tide – more devastating.

Very complicated, depend on many factors: Storm surge is a very complex phenomenon because it is sensitive to the slightest changes in storm intensity, forward speed, size (radius of maximum winds-RMW), angle of approach to the coast, central pressure (minimal contribution in comparison to the wind), and the shape and characteristics of coastal features such as bays and estuaries.
Surge + high tide

Mean sea level

Surge 15 ft.

17 ft. Storm tide

2 ft. Normal high tide
Storm surge

Z = 0

Pycnocline

300 times of sea level!
Approach the coast, D is shallow: sea level has to go up, surge amplified!

Gentle slope: amplify
Surge examples:

- The highest storm surge in record: 1899 Cyclone Mahina: 13 meters (43 feet) storm surge at Bathurst Bay, Australia (high tide);
- In the U.S., the greatest storm surge was generated by Hurricane Katrina: 9 meters (30 feet) high storm surge in Bay St. Louis, Mississippi, and surrounding counties. (Low elevation above sea level, larger impact)
Hurricane Katrina
Near peak strength: Aug 28, 2005

Formed: Aug 23; Dissipated: Aug 31

Highest: 175mph
Lowest pressure: 902mbar
• Damages: $81.2 billion (costliest Atlantic hurricane in history), the 6th strongest hurricane;
• Fatalities: greater than 1836 total;
• Areas affected: Bahamas, South Florida, Cuba, Louisiana (especially greater New Orleans), Mississippi, Alabama, Florida Panhandle, most of the eastern North America.
Aftermath of Katrina

Storm Surge video: NOAA National Weather Service:
https://www.youtube.com/watch?v=2GgUn2QTJtE&feature=emb_rel_end

Sea, Lake, and Overland Surges from Hurricanes (SLOSH)
3. Effects of rotation ($f \neq 0$) and Rossby radius of deformation

With $f=0$, what transient waves are available in the system?

What is the equilibrium state of the ocean after the waves propagation?

Critical thinking: what effects do you think $f$ will have on the transient waves and equilibrium state?
Effects of rotation \((f \neq 0)\) and Rossby radius of deformation

With a uniform rotation \((f\ \text{is assumed to be a constant})\), the equations of motion for the unforced, inviscid ocean are:

\[
\begin{align*}
\frac{dU}{dt} - fV &= -\frac{1}{\rho_0} \frac{\partial P}{\partial x}, \\
\frac{dV}{dt} + fU &= -\frac{1}{\rho_0} \frac{\partial P}{\partial y}, \\
\frac{dW}{dt} &= -\frac{1}{\rho_0} \frac{\partial P}{\partial z} - g. \\
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} &= 0.
\end{align*}
\]

**Assumptions:**

(i) \(f\ \text{constant};\)

(ii) \(\rho = \rho_0 = \text{constant};\)

(iii) \(\frac{\partial P}{\partial z} \ \text{Total P}\)

(iv) \(W = 0,\ \text{at } z=H;\)

(v) Background state:

\[U_0 = V_0 = W_0 = 0\]

\((Ro<<1, E<<1)\)
For small perturbations $u,v,w,p$ about the resting state, we have:

\[
\begin{align*}
U &= U_0 + u = u, \\
V &= V_0 + v = v, \\
W &= W_0 + w = w, \\
P &= P_0(z) + p, \quad \frac{\partial P}{\partial z} = -\rho_0 g. \\
\rho &= \rho_0.
\end{align*}
\]

The linearized first order equations for perturbation:

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + f u &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{align*}
\]
Apply boundary conditions

\[ Z=0, \quad p = \rho_0 g \eta, \quad w = \frac{\partial \eta}{\partial t} \]

\[ Z=H, \quad w = 0; \]

and vertically integrate the perturbation equations:

\[
\frac{\partial u}{\partial t} - f v = -g \frac{\partial \eta}{\partial x},
\]

\[
\frac{\partial v}{\partial t} + f u = -g \frac{\partial \eta}{\partial y},
\]

\[
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0.
\]
Following the same procedure as in the non-rotating case, write an equation in $\eta$,

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla^2 \eta + f H \zeta = 0,$$

Where

$$c^2 = gH, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Relative vorticity
Note: $f + \zeta$ is referred to as absolute vorticity;

Planetary vorticity

Relative vorticity

a) Non-rotating case ($f=0$):

$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla^2 \eta = 0$

Assume $\eta = \eta_0 \cos(kx - \omega t)$ (1-dimensional exp)

$\omega^2 = c^2 k^2$

$c^2 = gH$

Dispersion curve

Recall this is the dispersion relation for long-surface gravity waves when $f=0$!
Today’s class: dispersion relation

Previous: dispersion relation

Only has long surface gravity waves;

short waves: distorted by hydrostatic approximation

$$\frac{\partial P}{\partial z} = -\rho_0 g.$$
\[
\omega^2 = c^2 k^2 \quad \omega = \pm kc \quad f=0
\]
b) Rotating case ($f \neq 0$)

Assume wave form of solution
\[ \eta = \eta_0 e^{i(kx + ly - \omega t)}, \]
\[ u = u_0 e^{i(kx + ly - \omega t)}, \]
\[ v = v_0 e^{i(kx + ly - \omega t)}, \]

Substitute into the vertically-integrated perturbation equation for $\eta, u, v$

\[ \frac{\partial u}{\partial t} - f v = -g \frac{\partial \eta}{\partial x}, \]
\[ \frac{\partial v}{\partial t} + f u = -g \frac{\partial \eta}{\partial y}, \]
\[ \frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \]

let coefficient matrix = 0,

\[ \omega^2 = \kappa^2 c^2 + f^2, \quad \text{where} \quad \kappa^2 = k^2 + l^2 \]
\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} =
\]
\[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\]

\[\omega^2 = \kappa^2 c^2 + f^2,\]
What does $f$ do to the system?

$$\omega^2 = \kappa^2 c^2 + f^2,$$

Inertial gravity waves

(i) Long gravity waves become “dispersive”;

(ii) Long gravity waves do not have “zero” frequency anymore. Their lowest frequency is “$f$”, which has a period of a few days in mid latitude.
Adjustment with $f$ (1-dimensional exp)

Geostrophic balance

Equilibrium state

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} + f H \frac{\partial v}{\partial x} = 0$$

$$f v = g \frac{\partial \eta}{\partial x}$$
Solutions:

\[ \eta = \eta_0 \left[ -1 + \exp \left( -\frac{x}{a} \right) \right], \text{ for } x > 0, \]

\[ \eta = \eta_0 \left[ 1 - \exp \left( \frac{x}{a} \right) \right], \text{ for } x < 0, \]

where

\[ a = \frac{c}{f} \]

is Rossby radius of deformation.