Learning objectives: should be able to identify the observed features in subtropical ocean circulation, and understand their causes.

Specifics:
1. Observations: upper ocean circulation
2. Sverdrup relation, Sverdrup balance;
3. Westward intensification
1. Observed upper-ocean circulation

Schematic diagram
Current from geostrophic method

N. Atlantic

Dynamic height at 100db with reference to 700db.

Disparity in scales!

Interior: large

Western boundary: narrow
Westward intensification!

Pedlosky 1987

Fig. 1.1.2. Transport estimates of elements of the upper ocean circulation (waters warmer than 7°C) in the North Atlantic. Circled numbers, estimates of the number of Sverdrups carried in each branch of the circulation. (From Schmitz and McCartney 1993)
2: Ocean Interior Solution
Scale analysis, Sverdrup relation, Sverdrup balance

Development of Sverdrup theory:

Step 1: Derive a simplified vorticity balance for the interior ocean;
Step 2: Vertically integrate of the vorticity equation from step 1 under appropriate conditions, which yields a remarkable relationship between the vertically integrated meridional flow and the local value of the wind stress.

Scale analysis: What are the scales for large-scale motion in ocean interior? Ro and Ekman numbers?
Horizontal equations of motion:

\[
\frac{du}{dt} - fv = \frac{1}{\rho} \frac{\partial P}{\partial x} + A_H \nabla_H^2 u + \frac{1}{\rho} \frac{\partial \tau^x}{\partial z}, \tag{1a}
\]

\[
\frac{dv}{dt} + fu = \frac{1}{\rho} \frac{\partial P}{\partial y} + A_H \nabla_H^2 v + \frac{1}{\rho} \frac{\partial \tau^y}{\partial z}, \tag{1b}
\]

By considering $R_o << 1$ and $E_H << 1$, the equations of motion for the steady ocean circulation become:

(earlier classes: scale analysis for interior ocean, with $Lx=Ly=1000$km)
Nonlinear terms (advection), mixing terms, and time varying term are small & neglected.

Oceanic layers: surface mixed layer, and deeper ocean

\[
-fv = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial \tau^x}{\partial z}, \quad (2a)
\]

\[
f u = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{1}{\rho} \frac{\partial \tau^y}{\partial z} . \quad (2b)
\]
(a) Mixed layer:

\[
\begin{align*}
\rho f v &= \frac{\partial P}{\partial x} - \frac{\partial \tau^x}{\partial z}, \\
\rho f u &= -\frac{\partial P}{\partial y} + \frac{\partial \tau^y}{\partial z}.
\end{align*}
\]

(b) Below mixed layer:

\[
\begin{align*}
\rho f v &= \frac{\partial P}{\partial x}, \\
\rho f u &= -\frac{\partial P}{\partial y}.
\end{align*}
\]

Mixed layer. Performing \( \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \) to eliminate the \( P \) terms,

\[
\frac{\partial (\rho f u)}{\partial x} + \frac{\partial (\rho f v)}{\partial y} = \frac{\partial (\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y})}{\partial z}.
\]

Applying the Boussinesque approximation, we can use \( \rho_0 \) to replace \( \rho \) in the above equation. Re-arranging the above equation, we obtain

\[
\rho_0 f \frac{\partial u}{\partial x} + \rho_0 f \frac{\partial v}{\partial y} + \rho_0 \beta v = \frac{\partial (\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y})}{\partial z}.
\]
Using
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}, \]

\[ -\rho_0 f \frac{\partial w}{\partial z} + \rho_0 \beta v = \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right). \]

So in the mixed layer we have,

\[ \rho_0 \beta v = \rho_0 f \frac{\partial w}{\partial z} + \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right). \]  \hspace{1cm} (7)

They are:
- advection of planetary vorticity
- stretching term of water column,
- vorticity input from windstress.

Below the mixed layer: Sverdrup relation

\[ \beta v = f \frac{\partial w}{\partial z}. \]  \hspace{1cm} (8)

Linear Vorticity equation
\[ \beta v = f \frac{\partial w}{\partial z} \]

For a northward motion (NH),

\[ v > 0, \beta v = \frac{df}{dt} > 0 \]

\[ \frac{\partial w}{\partial z} > 0 \]

Potential vorticity conservation:

\[ \frac{f + \zeta}{H} \approx \frac{f}{H} = \text{constant} \]
Questions

Below the oceanic surface mixed layer, a water column moves southward in NH; how will the column change, stretching or compressed?

\[ \beta v = f \frac{\partial w}{\partial z} \]
Step 2: vertically integrate the vorticity equation

\[ \rho_0 \beta v = \rho_0 f \frac{\partial w}{\partial z} + \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) \]

\[ \rho_0 \beta \int_{-D-h_m}^{0} v \, dz = \rho_0 f \left[ w(0) - w(-D-h_m) \right] + \frac{\partial}{\partial x} \left[ \frac{\tau_y(0) - \tau_y(-D-h_m)}{\partial y} \right]. \]

Boundary conditions (Sverdrup 1947):

\[ w(0) = 0, \]
\[ w(-D-h_m) = 0, \]
\[ \tau_x(-D-h_m) = 0, \tau_y(-D-h_m) = 0. \]

At the surface, \( \tau_x(0) \) and \( \tau_y(0) \) are zonal and meridional wind stress.
Integration: \( \rho_0 \beta \int_{-D-h_m}^{0} v \, dz = \frac{\partial \tau^y_w}{\partial x} - \frac{\partial \tau^x_w}{\partial y} = \text{curl} \tau_w, \)

\[ \beta V_s = \frac{\text{curl} \tau_w}{\rho_0} \]

\[ V_s = \int_{-D-h_m}^{0} v \, dz, \quad \tau_w = (\tau^x_w, \tau^y_w) \]

NOTE: we used Boussinesque approximation; Approximate form.
As in the assumption, $w$ is zero at both the top and bottom of the ocean. According to the continuity equation:
\[ \int_{-D-h_m}^{0} u_x + v_y \, dz = 0, \text{ so } \nabla_H \cdot \mathbf{V}_s = 0. \]
\[ \mathbf{V}_s = \int_{-D-h_m}^{0} \mathbf{v} \, dz. \]
So the transport is horizontally non-divergent. By this non-divergent feature, we can introduce stream function $\psi$, so that
\[ U_s = -\frac{\partial \psi}{\partial y} \text{ and } V_s = \frac{\partial \psi}{\partial x}. \]
Then the Sverdrup balance can be written as:

\[ \beta \frac{\partial \psi}{\partial x} = \frac{\text{curl} \tau_w}{\rho_0}. \quad (10) \]
Schematic:
Surface winds

- Westerlies
- Subtropical deserts
- Northeast trades
- Southeast trades
- Equatorial rain-belt
- Storm tracks
Observations

Pedlosky 1987

15N-45N, STG region:

\[ \text{curl } \tau_w = - \frac{\partial \tau^x}{\partial y} < 0 \]
\[ \text{curl} \tau_w = -\frac{\partial \tau^x}{\partial y} \]

\[ \beta \frac{\partial \psi}{\partial x} = \frac{\text{curl} \tau_w}{\rho_0} \]

\[ U_s = -\frac{\partial \psi}{\partial y} \]

\[ V_s = \frac{\partial \psi}{\partial x} \]
Idealized winds:
\[ \tau_x = -\tau_0 \cos\left(\frac{\pi y}{L}\right), \]
\[ \tau_x = -\tau_0 \text{ at } 15^\circ\text{N}; \]
\[ \tau_x = \tau_0 \text{ at } 45^\circ\text{N}. \]
So the windcurl is
\[ \text{curl}\tau_w = -\frac{\partial \tau_x}{\partial y} = -\tau_0 \frac{\pi}{L} \sin\left(\frac{\pi y}{L}\right) = -a_0 \sin\left(\frac{\pi y}{L}\right), \]
where \( a_0 = \tau_0 \frac{\pi}{L} \).
The Sverdrup balance can be written as:
\[ \beta \frac{\partial \psi}{\partial x} = \frac{-a_0}{\rho_0} \sin\left(\frac{\pi y}{L}\right). \tag{2} \]
By applying boundary condition:
\[ \psi(x = L_E) = 0 \text{ at the eastern boundary}, \]
we obtain solution for \( \psi \):
\[ \psi(x) = \int_{L_E}^{x} \frac{-a_0}{\beta \rho_0} \sin\left(\frac{\pi y}{L}\right) dx = \frac{-a_0}{\beta \rho_0} \sin\left(\frac{\pi y}{L}\right)(x - L_E), \]

\[ \psi(x) = \frac{a_0}{\beta \rho_0} \sin\left(\frac{\pi y}{L}\right)(L_E - x). \] (3)
$\text{curl} \tau_w = -\frac{\partial \tau^x}{\partial y}$  \hspace{1cm} $\beta \frac{\partial \psi}{\partial x} = \frac{\text{curl} \tau_w}{\rho_0}$
Schematic diagram: wind-driven circulation in the ocean interior
Question

• Why is the upper ocean circulation in the northern hemisphere Subtropical ocean anticyclonic?

• What relationship have you derived and used to explain this circulation?
3. Westward intensification, western boundary current (WBC)

Scaling the equations of motion in WB region
Fig. 1.1.2. Transport estimates of elements of the upper ocean circulation (waters salinity than 7.00) in the North Atlantic. *Circled numbers,* estimates of the number of Sverdrups carried in each branch of the circulation. (From Schmitz and McCartney 1993)
Scaling in the western boundary region.
\( \rho = \text{const} \), steady state circulation: \( R_o << 1 \)

\[
-fu + \frac{1}{\rho} \frac{\partial P}{\partial x} = -\gamma u + A_H \nabla^2_H u + \frac{f u}{\rho H}, \quad (4a)
\]

\[
fu + \frac{1}{\rho} \frac{\partial P}{\partial y} = -\gamma v + A_H \nabla^2_H v, \quad (4b)
\]

\[
u_x + v_y = 0. \quad (4c)
\]

From \((4a)y - (4b)x\), we obtain

\[-\beta v = \gamma(u_x - u_y) - A_H \nabla^2_H (v_x - v_y) + \frac{\gamma u}{\rho H}. \]

Vertical integration:

\[-\beta Hv = \gamma [(Hv)_y - (Hu)_y] - A_H \nabla^2_H [(Hv)_y - (Hu)_y] + \frac{\gamma u}{\rho H}. \]

Denote \( Hu = U \) and \( Hv = V \), we have:

\[-\beta V = \gamma (V_x - U_y) - A_H \nabla^2_H (V_x - U_y) + \frac{\gamma u}{\rho H}. \]

Because \( u_x + v_y = 0 \) and thus \( U_x + V_y = 0 \), we introduce

\[
\bar{U} = -\frac{\gamma u}{y},
\]

\[
\bar{V} = \frac{\gamma u}{x},
\]