Learning objectives: understand internal waves, explain storm surge, and identify the effect of the Earth’s rotation

1. Internal waves; Barotropic and baroclinic modes;
2. Storm surges
3. Effects of rotation, Rossby radius of deformation
Previous classes: we discussed surface gravity waves. What assumptions did we make to isolate “surface” gravity waves?

*Constant density – homogeneous ocean*

**Observed ocean density:**

\[
\text{Sigma}(0) = (\text{density} - 1000) \text{ kg/m}^3
\]
Concepts: Vertical profiles of density, T, S

- Pycnocline
- Thermocline
- Haloine

Mixed layer

Thermocline, haloine and pycnocline
1. Internal gravity waves in density stratified ocean

Pycnocline
Internal gravity waves in density stratified ocean

a) A 2-layer model

\[ H_1 \quad 100-200 \text{m (mixed layer)} \]

Pycnocline

\[ H_2 \quad 3800 \text{m (deep ocean below Pycnocline)} \]
Barotropic and baroclinic modes

The 2-layer system has 2 vertical normal modes. Total Solution is the superposition of the two modes.

Barotropic mode: independent of $z$, which represents vertically-averaged motion.

Restoring force: gravity

Baroclinic mode: vertical shear flow, and vertically-integrated transport is zero. Restoring force is reduced gravity,

$$g' = \frac{g(\rho_2 - \rho_1)}{\rho_2} \sim 0.03 m/s^2$$
Barotropic mode

\[ Z = 0 \]

\[ H_1 \]

\[ H_2 \]

\[ \eta \]

\[ u_1 \rho_1 \]

\[ u_2 \rho_2 \]

Pycnocline

\[ C_0 = \sqrt{g(H_1 + H_2)} = \sqrt{gD} \sim 200 \text{m/s} \]

\[ h = \frac{H_2}{H_1 + H_2} \leq 1 \sim 1, \quad \frac{u_2}{u_1} \sim 1 \]
Baroclinic mode

\[ Z = 0 \]

\[ \eta \]

\[ u_1, \rho_1 \]

\[ u_2, \rho_2 \]

\[ h \]

\[ D \]

\[ C_1 = \sqrt{g' \frac{H_1 H_2}{H_1 + H_2}} \sim 2 - 3 \text{m/s} \]

\[ \frac{h}{\eta} = -\frac{g}{g'} \sim -300 \] (internal gravity waves)
A $1 \frac{1}{2}$ - layer model

Deep Ocean: infinitely deep and $\nabla \rho = 0$

The system has only one baroclinic mode. The barotropic mode is filtered out by the assumption $\nabla \rho = 0$.
2. Storm surge

Coastal shallow water: amplify. Why?
Basic dynamics:
(i) winds associated with storms or hurricanes pile the water up;
(ii) low sea level pressure at the storm center (minimal comparing with wind);
(iii) Shallow, gently sloping coastal region: intensify;
(iv) Overlapping with high tide – more devastating.

Very complicated, depend on many factors: Storm surge is a very complex phenomenon because it is sensitive to the slightest changes in storm intensity, forward speed, size (radius of maximum winds-RMW), angle of approach to the coast, central pressure (minimal contribution in comparison to the wind), and the shape and characteristics of coastal features such as bays and estuaries.
Surge + high tide
Storm surge

wind driven surge

Z=0

pressure surge

Pycnocline

D

300 times of sea level!
Approach the coast, D is shallow: sea level has to go up, surge amplified!

Gentle slope: amplify
Surge examples:

• The highest storm surge in record: 1899 Cyclone Mahina: 13 meters (43 feet) storm surge at Bathurst Bay, Australia (high tide);

• In the U.S., the greatest storm surge was generated by Hurricane Katrina: 9 meters (30 feet) high storm surge in Bay St. Louis, Mississippi, and surrounding counties. (Low elevation above sea level, larger impact)
Hurricane Katrina
Near peak strength:
Aug 28, 2005

Formed: Aug 23;
Dissipated: Aug 31

Highest: 175mph
Lowest pressure:
902mbar
• Damages: $81.2 billion (costliest Atlantic hurricane in history), the 6th strongest hurricane;
• Fatalities: greater than 1836 total;
• Areas affected: Bahamas, South Florida, Cuba, Louisiana (especially greater New Orleans), Mississippi, Alabama, Florida Panhandle, most of the eastern North America.
Aftermath of Katrina
3. Effects of rotation \((f \neq 0)\) and Rossby radius of deformation

With \(f=0\), what are the available transient waves in the system?

What is the equilibrium state of the ocean after the waves propagation?

Critical thinking: what effects do you think \(f\) will have on the transient waves and equilibrium state?
Effects of rotation ($f \neq 0$) and Rossby radius of deformation

With a uniform rotation ($f$ is assumed to be a constant), the equations of motion for the unforced, inviscid ocean are:

\[
\frac{dU}{dt} - fV = -\frac{1}{\rho_0} \frac{\partial P}{\partial x},
\]

\[
\frac{dV}{dt} + fU = -\frac{1}{\rho_0} \frac{\partial P}{\partial y},
\]

\[
\frac{dW}{dt} = -\frac{1}{\rho_0} \frac{\partial P}{\partial z} - g.
\]

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0.
\]

Assumptions:

(i) $f$ constant;

(ii) $\rho = \rho_0 = constant$;

(iii) $\frac{\partial P}{\partial z} = -\rho_0 g$.

(iv) $W = 0$, at $z=H$; bottom

(v) Background state:

$U_0 = V_0 = W_0 = 0$

(Ro<<1, E<<1)
For small perturbations \( u, v, w, p \) about the resting state, we have:

\[
\begin{align*}
U &= U_0 + u = u, \\
V &= V_0 + v = v, \\
W &= W_0 + w = w, \\
P &= P_0(z) + p, \\
\rho &= \rho_0.
\end{align*}
\]

The linearized first order equations for perturbation:

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v &= -1 \frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + f u &= -1 \frac{\partial p}{\partial y}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{align*}
\]
Apply boundary conditions

\( Z=0, \quad p = \rho_0 g \eta, \quad w = \frac{\partial \eta}{\partial t} \)

\( Z=H, \quad w = 0; \)

and vertically integrate the perturbation equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v &= -g \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + f u &= -g \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.
\end{align*}
\]
Following the same procedure as in the non-rotating case, write a single equation in $\eta$ alone,

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla^2 \eta + fH \zeta = 0,$$

Where

$$c^2 = gH, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Relative vorticity
Note: $f + \zeta$ is referred to as absolute vorticity;

**Planetary vorticity**

**Relative vorticity**

a) Non-rotating case ($f=0$):

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla^2 \eta = 0$$

Assume

$$\eta = \eta_0 \cos(kx - \omega t)$$

(1-dimensional exp)

Recall this is the dispersion relation for long-surface gravity waves when $f=0$!
\[ \omega^2 = c^2 k^2 \quad \omega = \pm kc \]
b) Rotating case ($f \neq 0$)
Assume wave form of solution

\[ \eta = \eta_0 e^{i(kx + ly - \omega t)}, \]
\[ u = u_0 e^{i(kx + ly - \omega t)}, \]
\[ v = v_0 e^{i(kx + ly - \omega t)}, \]

Substitute into the vertically-integrated perturbation equation for $\eta, u, v$, and let coefficient matrix=0,

\[ \omega^2 = \kappa^2 c^2 + f^2, \quad \text{where} \quad \kappa^2 = k^2 + l^2 \]
Math:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]
What does $f$ do to the system?

$$\omega^2 = \kappa^2 c^2 + f^2,$$

(i) Long gravity waves become “dispersive”;
(ii) Long gravity waves do not have “zero” frequency anymore. Their lowest frequency is “$f$”, which has a period of a few days in mid latitude.
Adjustment with $f$ (1-dimensional exp)

$\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} + fH \frac{\partial v}{\partial x} = 0$

Equilibrium state

Geostrophic balance $fv = g \frac{\partial \eta}{\partial x}$
Solutions:

\[ \eta = \eta_0 \left[ -1 + \exp \left( -\frac{x}{a} \right) \right], \text{ for } x > 0, \]

\[ \eta = \eta_0 \left[ 1 - \exp \left( \frac{x}{a} \right) \right], \text{ for } x < 0, \]

where

\[ a = \frac{c}{f} \]

is Rossby radius of deformation.