4.1 Wave concepts

When we think about waves usually we have a picture of undulations on the surface of the sea or lakes. They usually progress from a region of formation to a coast where they are dissipated or partly reflected (see Fig 1).

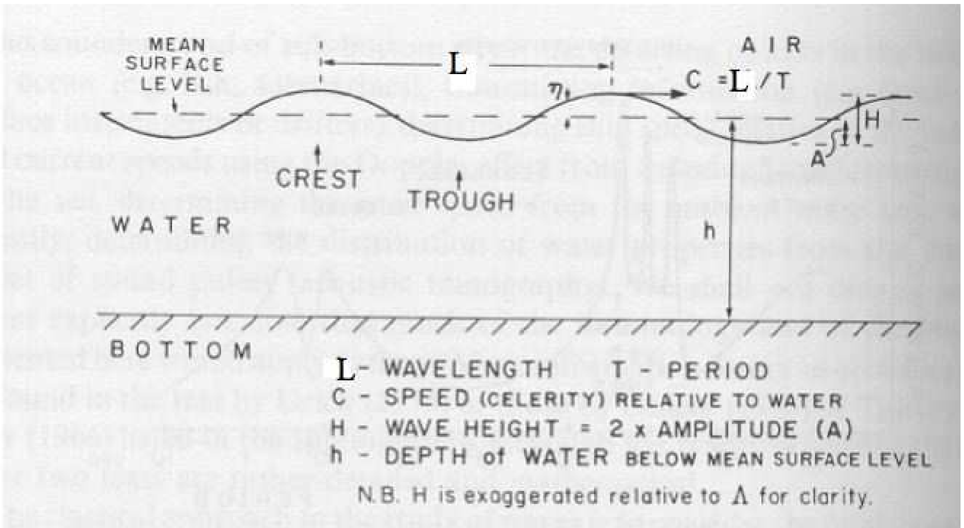

FIG. 1 *Terms related to ideal (sine or cosine) waves.*

Figure 1: Ideal sine or cosine waves.

Assuming that the waves at the sea surface are simple cosine waves.

Wave height (H) and amplitude (A): the vertical distance from trough to crest is called the "height" of a wave, which is twice of the "amplitude". The amplitude is the maximum displacement of the vertical oscillatory motion of the surface above or below the mean water level.

Wave length (L): The distance from crest to crest or from trough to trough. It is the distance that a wave travels in one period (T) of time.

Wave number (k): The number of waves that exist for a specified distance. The radian wave number $\frac{2\pi}{L}$ is often used.

Wave period (T): The time between two successive crests or troughs passing a fixed point.

Wave frequency (ω): The number of wave oscillations that take place in a specified time, say one second. The radian frequency $\frac{2\pi}{T}$ is often used.

Wave phase speed (C): The speed at which a particular part of the wave passes a fixed point, which is $C = L/T = \omega/k$.

Group speed; dispersion: In real ocean, waves are not pure sine waves but are a sum of sine waves with a range of wavelengths, corresponding periods and amplitudes. The group speed, C_g, at which the group travels is $\frac{\partial \omega}{\partial k}$. Fig 2 example. [P. & P. Fig 12.5). [see P. & P. for derivation].

Wave energy travels at the group speed, rather than phase speed. If the group speed changes with the change of frequency, we say the system is "dispersive", because the energy in the system will "disperse" over time. That is, waves at different frequencies travel at different speed. In contrast, if C_g is the same at all frequencies, the system is called non-dispersive.

Wave energy density. Total energy per unit area of sea surface is defined as $E = (\rho g A^2)/2 = (\rho g H^2)/8$ (joules m^{-2}).

Steepness. The ratio of H/L is called steepness.

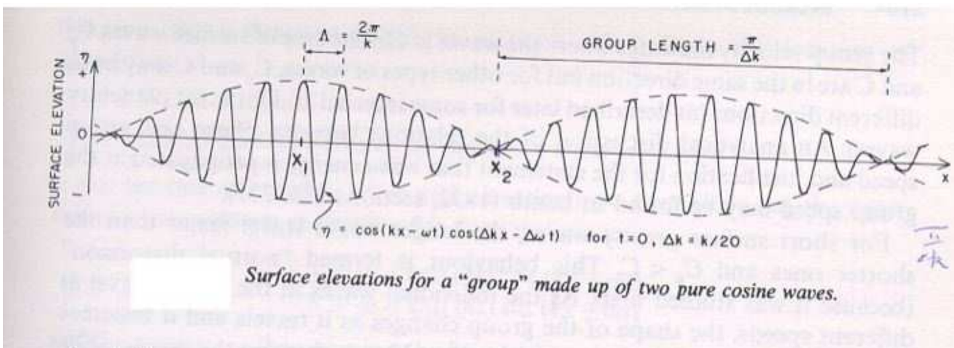

Figure 2:

4.2 Surface gravity waves: A homogeneous, inviscid ocean

The equations of motion we derived earlier can help us to understand two important aspects of the ocean circulation:

(i) the transient response of the ocean to changes in the surface forcing;

(ii) the mean ocean circulation driven by the surface winds and buoyancy effects.

We will discuss both of these aspects in this course, beginning with the transient ocean response–waves.

We will first discuss how the ocean will respond when there is a change in the forcing field. To this end a valuable insight into the adjustment of the ocean to changes in forcing comes from a study of linear ocean waves: Recall from the previous class that the Rossby and Ekman numbers (R_o and E_x) are small in the ocean interior, and the interior ocean can be viewed as "linear" and inviscid. Since Coriolis forcing is very important to large scale motion, we wish to isolate its effect. One approach to isolate this effect is to "eliminate" it from the equations first. Then see what it does to the ocean by adding it in.

The class will focus on the concepts and physics rather than detailed math, therefore we only briefly go through the mathematical procedures during our discussion. First, we will obtain the dispersion relation, and see what types of waves are available in our simplified system.

The equations of motion in a homogeneous, inviscid ocean by ignoring the Coriolis force are:

$$\frac{dU}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}, \tag{1a}$$

$$\frac{dV}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}, \tag{1b}$$

$$\frac{dW}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - g. \tag{1c}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0. \tag{1d}$$

Let's assume the equilibrium state of the ocean is at the state of rest. That is:
$U_0 = V_0 = W_0 = 0$,
$P_0 = P_0(z)$,
$\rho = \rho_0$ (constant–homogeneous).
For small perturbations u, v, w, p about the resting state, we have:
$U = U_0 + u = u$,
$V = V_0 + v = v$,
$W = W_0 + w = w$,
$P = P_0(z) + p$,
$\rho = \rho_0$.
For the background state, the above equations of motion become:

$$-\frac{1}{\rho_0}\frac{\partial P_0}{\partial z} - g = 0. \tag{2}$$

For the perturbations, to the *first order* (ignore the higher order terms), in a non-rotating frame ($f = 0$), we have:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{3a}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{3b}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}. \tag{3c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3d}$$

From the above equations, we write a single equation in p alone:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0.$$

Boundary conditions:

- At the ocean bottom $z = -D$, we have $w = 0$. [Assume flat ocean bottom where D=constant].
- At surface, $z = \eta$, we require any fluid parcels at the surface to remain at the surface (material boundary condition). For such fluid parcels, $z - \eta = 0$ all the time. So

  $d(z - \eta)/dt = 0$, and

  $w = \frac{dz}{dt} = \frac{\partial \eta}{\partial t}$ (ignore nonlinear terms).
- At $z = \eta$, $P = P_0 + p = 0$, (zero pressure at the ocean surface, ignore the surface atmospheric pressure). Then $p = -P_0 = \rho_0 g \eta$.

  Since $\eta << D$ so we can take $z = \eta = 0$ so the condition is:

  At $z = 0$, $p = \rho_0 g \eta$.

Now, the ocean is bounded in z direction but open in x and y directions. So we can assume free wave form in x and y, but leave the vertical structure undetermined.

Assume a traveling wave-like solutions of the form:
$\eta = \eta_0 cos(kx + ly - \omega t)$,
$p = p_i(z) cos(kx + ly - \omega t)$,
Substituting these solutions to the single equation in p alone, we have:
$\frac{\partial^2 p_i}{\partial z^2} - (k^2 + l^2) p_i = 0$.
Thus,
$p_i(z) = A sinh(\kappa z) + B cosh(\kappa z)$,
where $\kappa^2 = k^2 + l^2$.
By applying bottom boundary condition (at $z = -D$, $w = 0$ and thus $\frac{\partial p}{\partial z} = 0$ from equation (3c)), and surface boundary condition $p = \rho_0 g \eta$ at $z = 0$, we obtain,

$$p = \frac{g \rho_0 \eta_0 cos(kx + ly - \omega t) cosh \kappa (z + D)}{cosh(\kappa D)}. \tag{4}$$

From equation (3c) we have

$$w = \frac{\kappa g \eta_0 sin(kx + ly - \omega t) sinh \kappa (z + D)}{\omega cosh(\kappa D)}. \tag{5}$$

Now let equations (4) and (5) satisfy boundary condition we derived above $w = \frac{\partial \eta}{\partial t}$ at $z = 0$, we obtain the dispersion relation:

$$\omega^2 = g \kappa tanh(\kappa D). \tag{6}$$

See Fig 3 for the dispersion relation diagram.

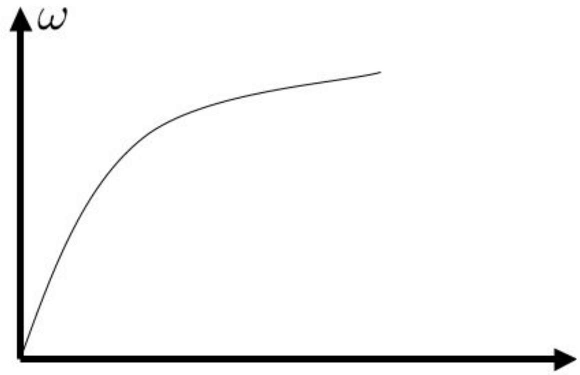

Figure 3: Dispersion relation for surface gravity waves.

- *Long waves.* For long waves such that $L - -\frac{1}{\kappa} >> D$ and thus $\kappa D << 1 ===> tanh(\kappa D) = \kappa D$. Equation (6) becomes:

  $\omega^2 = gD\kappa^2$.

  Thus, the wave speed $C^2 = gD$. $C_g = \frac{\partial \omega}{\partial \kappa} = \pm\sqrt{gD}$, which does not change with frequency, and thus is non-dispersive.

  Long surface gravity waves are non-dispersive. These waves are called gravity waves because their restoring force is "gravity". They can propagate in all directions of a horizontal plane.

  From equations (3) we can see that, for an initial perturbation in p, the ocean will respond in transient motion by radiating gravity waves in all directions of a horizontal plane. After the gravity waves radiate out, that is the $\frac{\partial}{\partial t}$ terms equal 0, the ocean reaches a state of rest. After an initial perturbation, the ocean adjusts to an equilibrium state under gravity in a non-rotating ocean. A good example is we throw a big stone in a calm pond. The initial perturbation will cause gravity waves to radiate out in circles, after these waves radiate out, the pond is calm again.

  Vertical structure. Since $(\kappa D) << 1$, then $cosh(\kappa D) - 1$, and

$$p = g\rho_0\eta_0 cos(kx + ly - \omega t), \tag{7}$$

$$\frac{\partial w}{\partial t} = 0. \tag{8}$$

  and thus $w = constant$.

  Independent of depth. They feel the bottom of the ocean.

- *Short waves.* For the short waves with $L - \frac{1}{\kappa} << D$ and thus $(\kappa D) >> 1$, the dispersion relation becomes:

$\omega^2 = g\kappa$.

$C = \frac{\omega}{\kappa}$ and $C_g = \frac{\partial \omega}{\partial \kappa}$ is a function of ω and thus they are dispersive.

By simplifying their solutions by the dispersion relation, we have

$$p = g\rho_0\eta_0 cos(kx + ly - \omega t)e^{\kappa z}, \tag{9}$$

$$w = \frac{\kappa g\eta_0 sin(kx + ly - \omega t)e^{\kappa z}}{\omega} \tag{10}$$

Short waves are trapped to the upper ocean to a region of depth κ^{-1}. They do not feel the ocean bottom.

Frequency range for gravity waves: 0–infinity in a non-rotating system (f=0), and at low frequency, wave number is low and thus wavelength is long. As we shall see, in a rotating system these long waves are modified and thus gravity waves exist only in the period band of seconds–days.

The wave energy propagation is different from water particle movement. In fact, to the lowest order, the water particles for short waves are circular and for long waves are elliptical. [Need to solve the solutions to know the details]. If we consider the higher order effects the orbits of particles are not quite closed; for short waves there is a net flow in the direction of travel and this net transport is called *Stokes drift*. However, this effect is small (a small fraction of the orbital speed).

4.3 Refraction and breaking in shallow water

(i) Refraction. As we just discussed, small amplitude long waves all travel at the same speed in water of a given depth D ($C = \sqrt{gD}$). But if the bottom depth is changing their direction of travel may change. As waves move into shallow water their period remains constant but C decreases and therefore wavelength decreases. [$\omega = \kappa C$. If ω is constant and C decreases, then κ has to increases and thus wavelength ($L = \frac{2\pi}{\kappa}$) decreases.] If a series of parallel-crested waves approaches at an angle to a straight shoreline (Fig 4) over a smooth sea bottom which shoals gradually, they progressively change direction as the end of the wave nearer to the shore. As a result, the waves become more parallel to the shore by the time that they pile up as surf. *The change in direction associated with the change of speed is called refraction.* The refraction pattern will be more complicated if the ocean bottom has complicated topography.

(ii) Breaking. As waves move in-shore and slow down, not only does the wavelength decrease but also the wave height changes. *In a steady state, the energy flux (the product of the wave energy per square meter times the group speed C_g) is conserved as the waves move in-shore (until they break).*

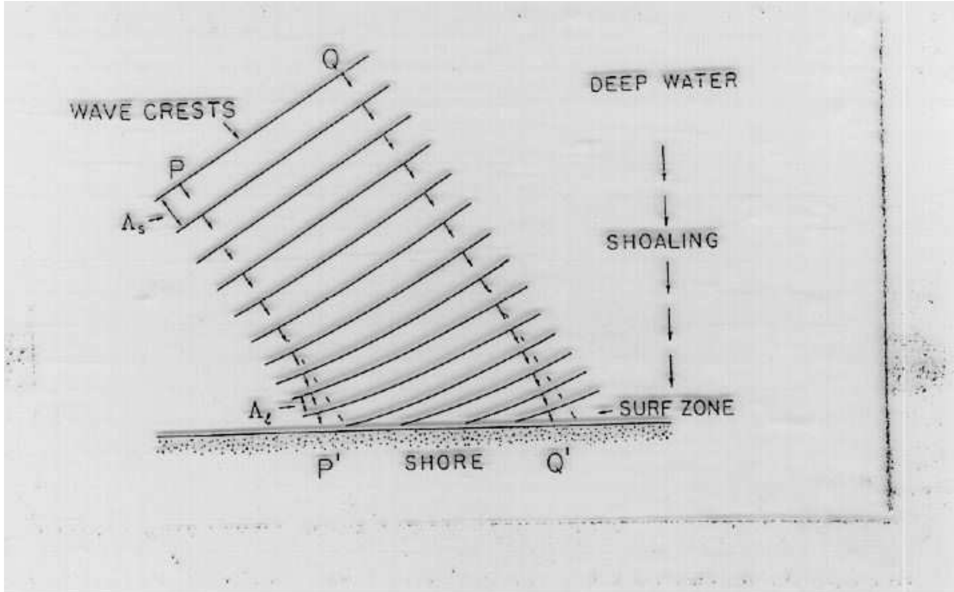

Figure 4: Wave regraction.

Energy flux $E_f = \rho g H^2 C_g/8$=constant. For the long waves, as it moves in-shore, C_g decreases and wavelength decreases (since frequency does not change), and thus H increases. Therefore, the waves become steeper and steeper. As the steepness goes to 1/12 in real ocean (1/7 in theory), the waves break (Fig 5).

4.4 Swells and Tsunamis

Swell: waves generated by persistent wind forcing. Large amplitudes and propagate for a long distance.

Tsunamis: Long water-waves usually generated by sea bottom movement associated with earthquakes. Sometimes called "seismic sea wave"; but they can also be caused by Meteorite. The effects are usually not widespread, but Tsunamis generated by seismic activity on one side of the Pacific have caused devastation on the other side, as in the case of the tsunami generated by the 1960 earthquake in Chile which caused serious damage there and also in Japan nearly 20,000km away. The recent December 2004 tsunami over Asia was devastating.

4.5 Internal gravity waves: density stratified ocean.

In the above, we discussed the waves in a homogeneous, inviscid ocean. We obtain gravity waves "either surface trapped" (short waves), or "depth independent" whose perturbation is large at the surface (long waves). We call these waves "surface gravity waves". In reality, the ocean is stratified rather than homogeneous. By considering the "stratification" of the ocean, we obtain "internal gravity waves".

a) A two-layer model.

The simplest case to represent the stratified ocean is to view the ocean as a two-layer

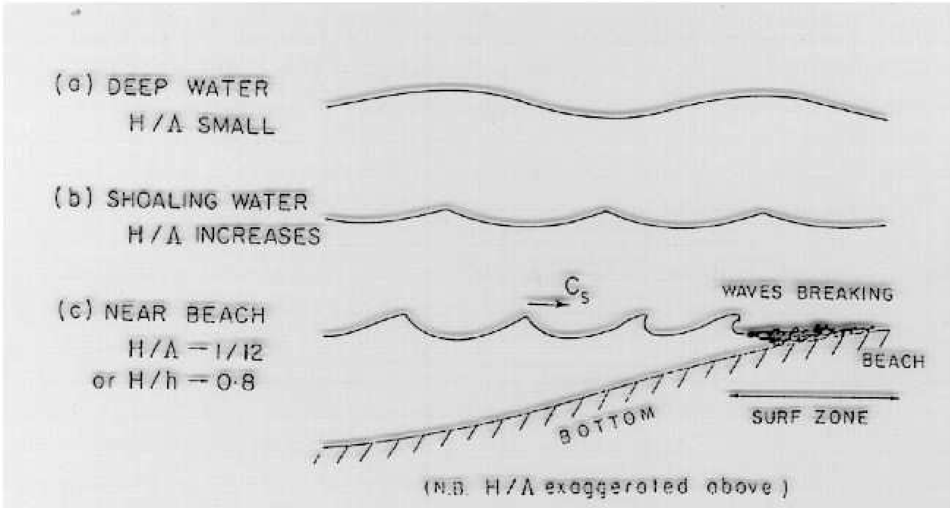

FIG. 12.9 *Shape of waves: (a) in deep water, (b) in shoaling water, (c) close to beach.*

Figure 5: Wave breaking.

system. Fig 6. This is "reasonable" because there is a well mixed surface layer in the real ocean, which is separated from the deep ocean by a sharp pycnocline. Usually, mixed layer depth $H_1 = 100m - 200m$, and the depth beneath the thermocline $H_2 = 3800m$. The "interface" in the two layer model represents the "pycnocline".

This system has two vertical modes: One barotropic mode and one baroclinic mode. The barotropic mode represents the vertically-averaged motion. The restoring force is gravity "g". The baroclinic mode has vertical shear and does not contribute to the vertically-integrated transport. The restoring force is "reduced gravity", $\frac{g'}{g} = 0.003$, and $g' = \frac{g(\rho_2 - \rho_1)}{\rho_2} = 0.03ms^{-2}$.

I will skip the derivation and just give the phase speeds for the two modes, respectively.

For the barotropic mode $C_0 = \sqrt{g(H_1 + H_2)}$.

For the baroclinic mode $C_1 = \sqrt{g' \frac{H_1 H_2}{H_1 + H_2}}$.

- For the barotropic mode,

  $\frac{h}{\eta} = \frac{H_2}{H_1 + H_2} \leq 1$,

  and $\frac{u_2}{u_1} \leq 1$.

  Because $H_2 \gg H_1$, η is close to h and u_2 is close to u_1. This is why we call it "barotropic mode". It is approximately depth independent. $\int_{-(H_1+H_2)}^{0} u dz \neq 0$.

  For $H_1 + H_2$=4000m, $C_0 = 200m/s$. Very fast. They can cross the Pacific within 2 days. It is surface gravity wave.

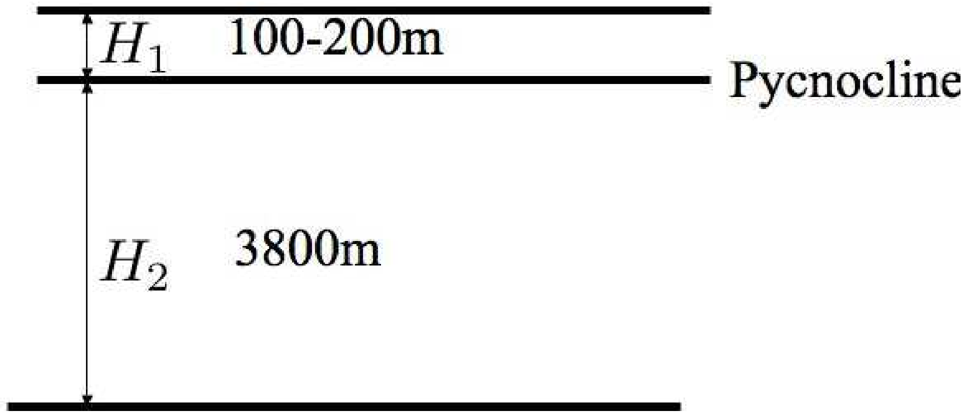

Figure 6: A two-layer model.

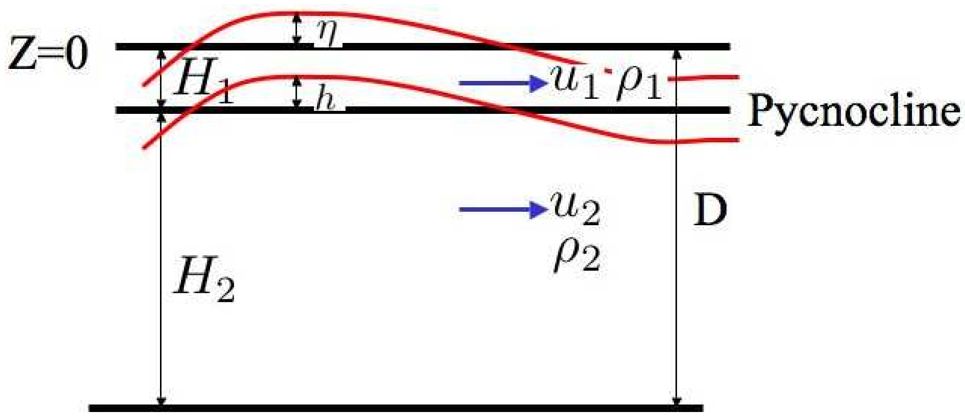

Figure 7a: Barotropic mode.

- For the baroclinic mode, $\frac{h}{\eta} = -\frac{g}{g'} = -300$.

  So $h = -300\eta$. If surface raises 1 cm, η=1cm, then the thermocline will deepen by 3m!! Since surface perturbation is much less than the variation of the "thermocline", we call these waves "internal gravity waves". Velocities for the baroclinic mode have opposite signs in the two layers. This mode, however, does not contribute to the total vertically integrated transport.

  $\int_{-(H_1+H_2)}^{0} udz = 0$.

  Its structure is vertically dependent, shear flow. $C_1 = 2 - 3m/s$ which is 100 times slower than the surface gravity wave.

  b) A $1\frac{1}{2}$-layer, reduced gravity model.

  As we have seen above, gravity waves associated with the barotropic mode are very fast. If we are only interested in the slower baroclinic wave, we wish to eliminate

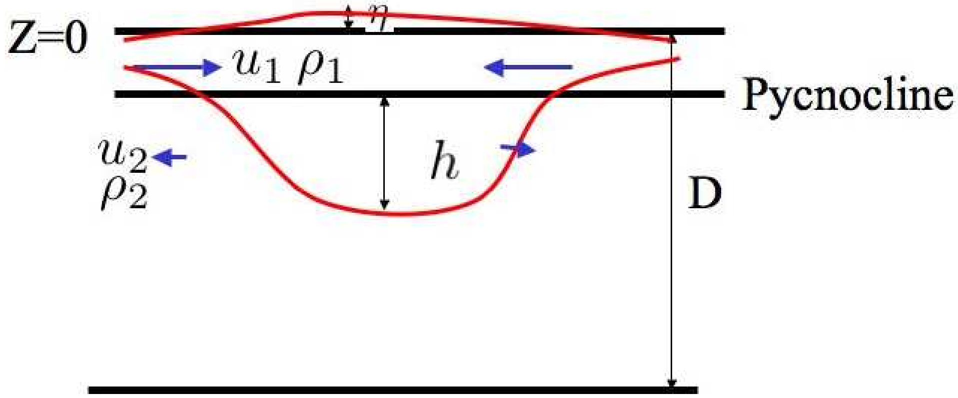

Figure 7b: Baroclinic mode.

the barotropic mode. One way to do so is to assume "reduced gravity" condition, which assumes that the deep ocean (below the surface mixed layer) is motionless. This assumption eliminates the barotropic mode, which is depth-independent. [NOTE: Recall that in the "geostrophic method" section, we assume a reference level - the level of no motion. We argued that this assumption filtered out the barotropic mode.]

Therefore, in a $1\frac{1}{2}$-layer system, we only have one baroclinic mode. Fig 8 Schematic diagram showing the baroclinic mode in a $1\frac{1}{2}$-layer system.

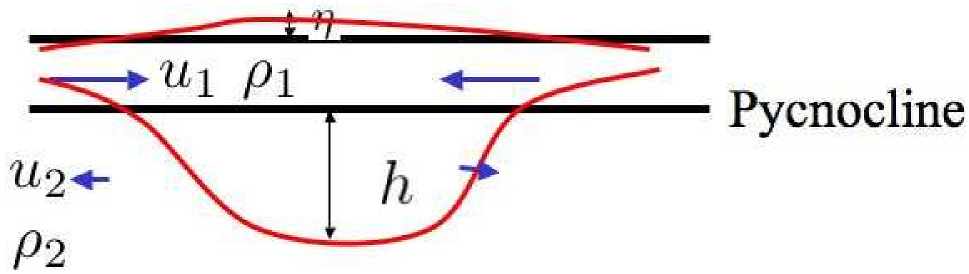

Figure 8: The baroclinic mode in the $1\frac{1}{2}$-layer model.

As discussed in the 2-layer model, the baroclinic mode has a large-amplitude perturbation in the thermocline, and a small-amplitude perturbation at the sea surface. Therefore, it represents internal waves. The "one active layer" above the "inert deep ocean" represents the oceanic "surface mixed layer", and the motion of the mixed layer bottom represents the movement of the thermocline. In this system, variation of the layer thickness results primarily from the thermocline motion. The baroclinic mode speed of gravity is: $c = \sqrt{g'H}$.

4.6 Effects of rotation, Rossby radius of deformation

In the above, we have considered the adjustment under gravity in a non-rotating ocean. In this system, after the transient gravity waves radiate out, the ocean reaches a equilibrium state: the state of rest. We eliminated the rotation effects by setting Coriolis parameter $f = 0$.

Now let's add "f" back into the equations, and see what it does to the system.

With a uniform rotation (f is assumed to be a constant), the equations of motion for the unforced, inviscid ocean are:

$$\frac{dU}{dt} - fV = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}, \tag{11a}$$

$$\frac{dV}{dt} + fU = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}, \tag{11b}$$

$$\frac{dW}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - g. \tag{11c}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0. \tag{11d}$$

Assumptions:

(i) f=constant (i.e. the f-plane approximation);

(ii) homogeneous ocean==> $\rho = \rho_0$=constant;

(iii) hydrostatic approximation ($\frac{\partial P}{\partial z} = -\rho_0 g$); This approximation eliminates the "short gravity waves", leave only the long waves in the system, as discussed in the previous class. As we recall, the long waves have a structure that is "independent of depth".

(iv) flat bottom, depth H at where $W = 0$.

(v) Linearize about a resting basic state ($U_0 = V_0 = W_0 = 0$)

For small perturbations u, v, w, p about the resting state, we have:

$U = U_0 + u = u,$

$V = V_0 + v = v,$

$W = W_0 + w = w,$

$P = P_0(z) + p,$

$\rho = \rho_0.$

To the *first order*, we have:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{12a}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{12b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{12c}$$

By applying hydrostatic approximation at $z = 0$, $P = P_0 + p = p = \rho_0 g\eta$ (note: p is independent of depth), vertically integrate equation (12c) and using surface boundary condition $w = \frac{\partial \eta}{\partial t}$, bottom boundary condition $w = 0$, we obtain:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}, \tag{13a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{13b}$$

$$\frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0. \tag{13c}$$

Figure 9.

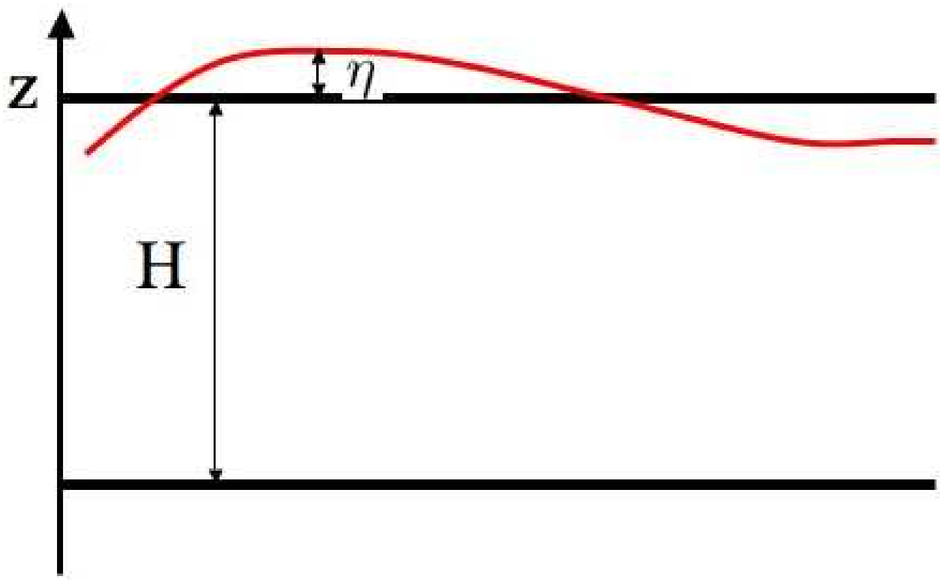

Figure 9: Schematic diagram showing the surface perturbation for the shallow water equation.

As in the non-rotating case, we combine the above three equations to write a single equation in η alone.

$$\frac{\partial^2 \eta}{\partial t^2} - c^2\nabla^2\eta + fH\zeta = 0, \tag{14}$$

where $c^2 = gH$ is phase speed squared, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is called *relative vorticity*, f is planetary vorticity, and $\zeta + f$ is called absolute vorticity.

(a) Non-rotating case ($f = 0$). Fig 10. Schematics for gravity waves radiation. Let's look at the simple, one-dimensional case in x direction. If there is no f, equation (14) becomes,

$\frac{\partial^2 \eta}{\partial t^2} - c^2\nabla^2\eta = 0.$

Assume $\eta = \eta_0 cos(kx - \omega t)$, then we obtain from the above equation:

$-\omega^2 + c^2k^2 = 0$, and $\omega^2 = c^2k^2$, thus

$\omega = \pm kc$. That is, gravity waves radiate out in $-x$ and $+x$ directions with a speed of $c = \sqrt{gH}$. After these waves radiate out, the sea surface flatens and thus $\eta = 0$.

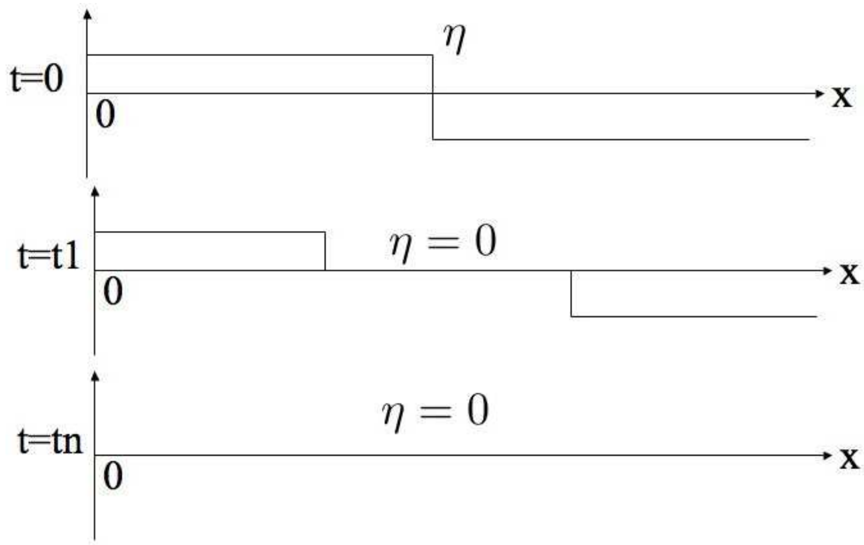

Figure 10: Schematic diagram showing the oceanic adjustment in the non-rotating case.

(b) Rotating case ($f \neq 0$). For one dimensional case, equation (14) becomes:
$\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} + fH\frac{\partial v}{\partial x} = 0$.

After gravity waves radiate out, that is, $\frac{\partial^2 \eta}{\partial t^2} = 0$ in the above equation, we obtain an equilibrium state:

$-c^2 \frac{\partial^2 \eta}{\partial x^2} + fH\frac{\partial v}{\partial x} = 0$, and thus:

$-g\frac{\partial \eta}{\partial x} + fv = 0$, or

$fv = g\frac{\partial \eta}{\partial x}$.—Geostrophic balance.

Solutions to η [for derivation see Gill 7.2.1 P192] is:

$$\eta = \eta_0[-1 + exp(-\frac{x}{a})], for x > 0, \tag{15a}$$

$$\eta = \eta_0[1 - exp(\frac{x}{a})], for x < 0, \tag{15b}$$

where $a = \frac{c}{f}$ is called the Rossby radius of deformation. As we can see from the η solution, sea level is deformed within the distance of "a", and the geostrophic currents are confined within the range of the Rossby radius. Outside this range, v is close to 0. Therefore, Rossby radius is a natural scale for the large-scale ocean circulation. [For the first baroclinic mode with c_1–3 m/s, a_1 is about 30km at 45°N where $f = 10^{-4}/s$.]

Now we can see that, with effects of rotation, the equilibrium state of the ocean is in "geostrophic balance", rather than the "state of rest" as in the non-rotating case.

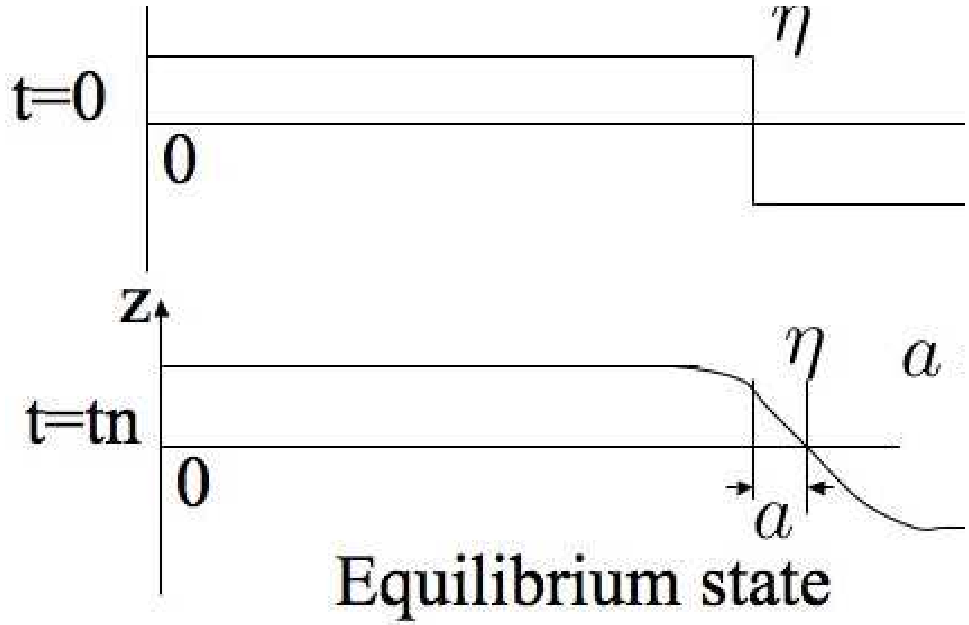

Figure 11: Schematic diagram showing the equilibrium state in the rotating case.

We just saw that rotation can change the "equilibrium" state of the ocean, relative to the non-rotating case. Next, we will consider how rotation modifies the gravity waves in the system.

By assuming the wave form of:

$$\eta = \eta_0 expi(kx + ly - \omega t), \tag{16a}$$

$$u = u_0 expi(kx + ly - \omega t), \tag{16b}$$

$$v = v_0 expi(kx + ly - \omega t), \tag{16c}$$

substituting them into equations (11), and let the coefficient matrix equal "0", we obtain the dispersion relation

$$\omega^2 = \kappa^2 c^2 + f^2, \tag{17}$$

where $\kappa^2 = k^2 + l^2$.

(i) First, the long waves become "dispersive", since $C_g = \frac{\partial \omega}{\partial \kappa}$ is a function of ω and thus κ.

(ii) Second, the long gravity waves do not have "zero" frequencies anymore. Their "lowest" frequency is at "f", which has a period ($T = \frac{2\pi}{\omega}$) of a few days. That is, the longest period for gravity waves is "days". Their period range is "seconds–days". They are considered "high" frequency in "oceanography", since Rossby waves have periods of months to years.

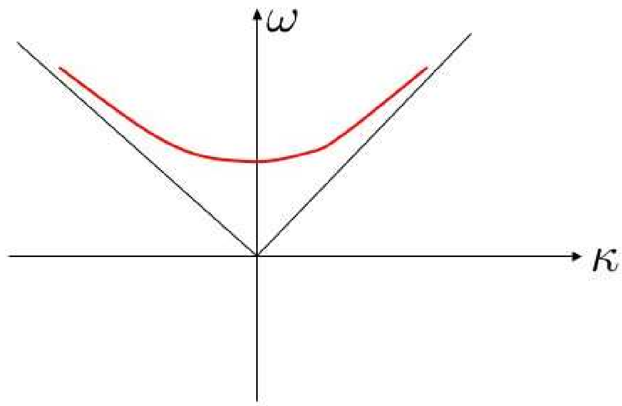

Figure 12: Schematic diagram showing the dispersion relation for gravity waves with the influence of rotation.

4.7 Waves in an ocean with a varying f: Rossby waves

Up to now, we have considered the adjustment under gravity in a non-rotating ocean with $f = 0$, and in a rotating ocean with $f = constant$. In these systems, gravity waves exist and they are modified by rotation in the rotating case. Since the earth is rotating and Coriolis parameter $f = 2\Omega sin\phi$, f in fact varies with latitude ϕ. Therefore, we can write

$f = f_0 + \beta y$,

where f_0 is the Coriolis parameter at latitude ϕ_0, say $\phi = 45°$N, and y is meridional distance from latitude ϕ to ϕ_0. It is referred to as mid-latitude "β-plane approximation". $f = f_0$ is referred to as "f-plane approximation".

Figure 13 Schematic diagram for β-plane.

A few relevant concepts. We often refer to:

f as *planetary vorticity*,

$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ as *relative vorticity*,

$f + \zeta$ as *absolute vorticity*, and

$\frac{\zeta + f}{H}$ as *potential vorticity*, where H is the depth of water column.

Again, we use the simple shallow water equation to demonstrate what β does to the system. The equations are the same as those in the f-plane case discussed in the previous class, except that $f = f_0 + \beta y$ here rather than $f = f_0 = constant$.

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}, \tag{18a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{18b}$$

$$\frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0. \tag{18c}$$

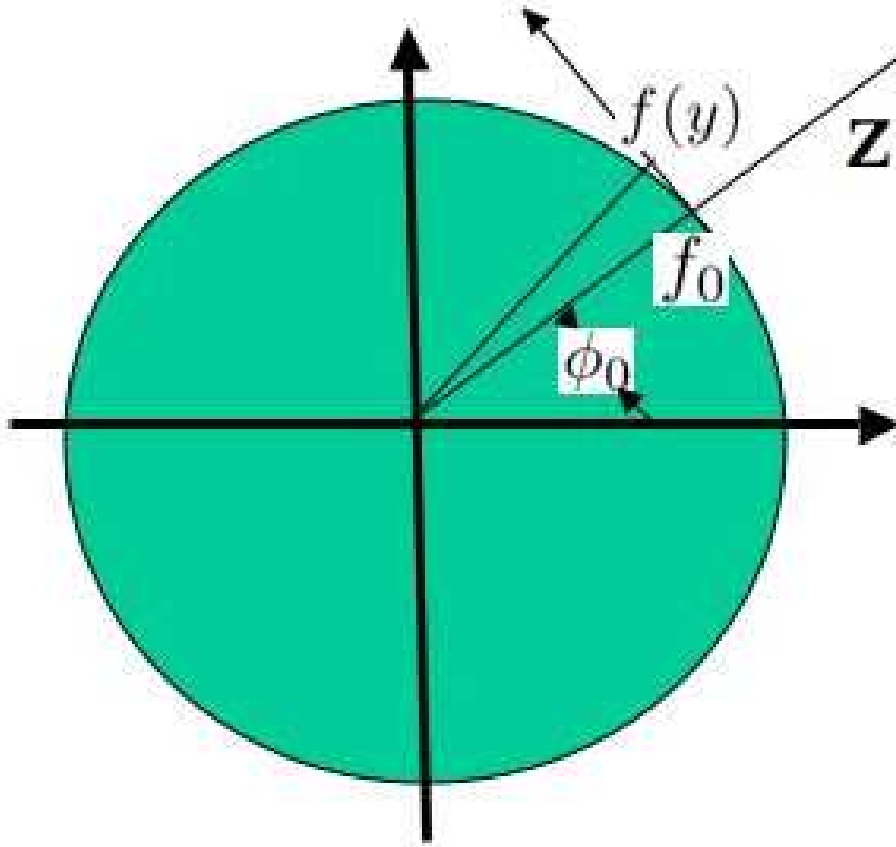

Figure 13: Schematic diagram for β-plane.

We combine the above three equations to write a single equation in v alone, and assume periodic wave solution $v - -e^{-i\omega t}$, we obtain:

$$\omega^3 v + \omega c^2 \frac{\partial^2 v}{\partial x^2} - f_0^2 v\omega + c^2 \beta \frac{\partial v}{\partial x} i + \omega c^2 \frac{\partial^2 v}{\partial y^2} = 0, \tag{19}$$

where $c = \sqrt{gH}$ is phase speed for long surface gravity waves. Assuming wave form: $v - -e^{i(kx+ly-\omega t)}$ in equation (19) and re-arrange the equation, we obtain the dispersion relation:

$$(k + \frac{\beta}{2\omega})^2 + l^2 = \frac{\omega^2 - f_0^2}{c^2} + \frac{\beta^2}{4\omega^2}. \tag{20}$$

Let's look at the simple case with $l = 0$, and we shall see what waves are available in the system. The dispersion relation becomes:

$$(k + \frac{\beta}{2\omega})^2 = \frac{\omega^2 - f_0^2}{c^2} + \frac{\beta^2}{4\omega^2}. \tag{21}$$

(i) For high frequency waves with large ω, the terms with ω in the denominator can be ignored. Thus,
$k^2 = \frac{\omega^2 - f_0^2}{c^2}$, or

$$\omega^2 = k^2c^2 + f_0^2. \tag{22}$$

These are long surface gravity waves under the influence of "f"–they are also called *Inertial gravity waves.*

(ii) For low frequency waves with small ω, we can drop ω^2 term. Then we have:
$(k + \frac{\beta}{2\omega})^2 = \frac{\beta^2}{4\omega^2} - \frac{f_0^2}{c^2}$. Rewrite it,

$$k = -\frac{\beta}{2\omega} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\omega^2} - 4\frac{f_0^2}{c^2}} = -\frac{\beta}{2\omega}(1 \pm \sqrt{1 - 4\frac{f_0^2\omega^2}{c^2\beta^2}}). \tag{23}$$

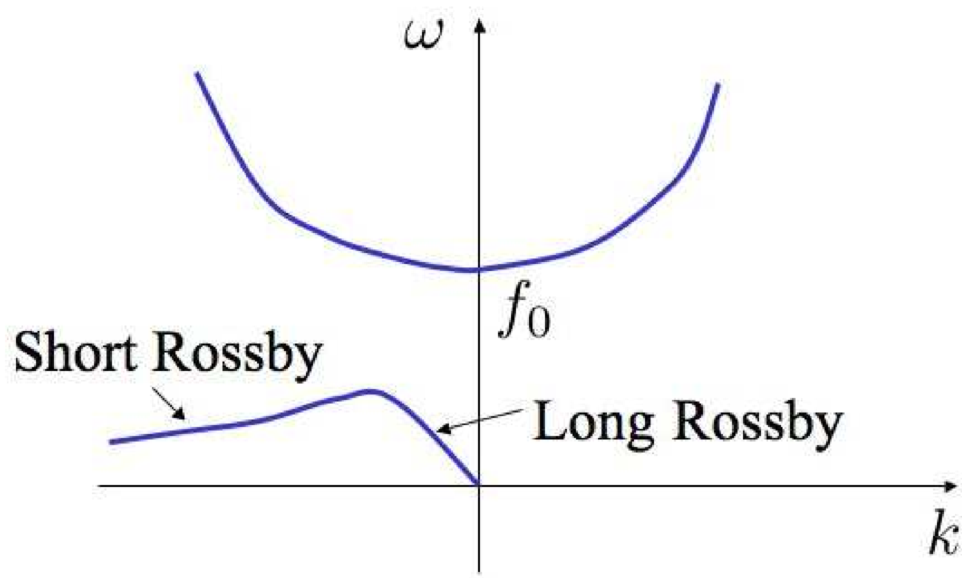

Figure 14: Schematic diagram showing the dispersion relation for inertial gravity waves and Rossby waves.

To further understand the wave properties, we need to obtain simplified results. Since $\frac{f_0^2\omega^2}{c^2\beta^2}$ is a small number given that ω is small, the above equations can be approximated by:
$k = -\frac{\beta}{2\omega}[1 \pm (1 - 2\frac{f_0^2\omega^2}{\beta^2c^2})]$.

Choosing the "-" sign we have,
$k = -\frac{\beta}{2\omega}(1 - 1 + 2\frac{f_0^2\omega^2}{\beta^2 c^2}) = -\frac{\omega f_0^2}{\beta c^2}$, or

$$\omega = -\frac{\beta c^2}{f_0^2} k = -c_r k. \tag{24}$$

- Since $c_g = \frac{\partial \omega}{\partial k} = -c_r$ is independent of ω, these waves are non-dispersive and both phase ($c_p = \frac{\omega}{k} = -c_r$) and group velocity (energy c_g) propagate westward. From the dispersion relation we can see these waves are: long Rossby waves.

- No matter f is positive (Northern Hemisphere) or negative (Southern Hemisphere), long Rossby waves propagate westward, both for energy and phase.

- Long Rossby wave speed decreases with the increase of latitude: $c_g = c_p = -c_r = -\frac{\beta c^2}{f_0^2}$. As the absolute value of f_0 increases poleward, Rossby waves speed decreases.

Choose the "+" sign and ignore the second order ω term:

$$\omega = -\frac{\beta}{k}. \tag{25}$$

Dispersive since $c_g = \frac{\partial \omega}{\partial k} = \frac{\beta}{k^2}$ is a function of k and thus ω. $c_p = \frac{\omega}{k} = -\frac{\beta}{k^2}$. These waves are *short Rossby waves: Their group velocity propagates eastward but phase propagates westward.* These can be seen from the dispersion curve. Short Rossby waves are very short so that mixing acts strongly on these waves. They are easily killed by mixing and thus very hard to propagate far away from the forcing region. Also, in real open ocean they are not strongly excited by the "large-scale" winds because their wavelengths are too small. Near the western boundary, these waves are killed by the strong mixing there and thus contribute to the WBC.

- **Existence of Rossby waves are due to β: the variation of planetary vorticity (f) with latitude (ϕ), and $\beta = \frac{\partial f}{\partial y}$. If we set $\beta = 0$ then the dispersive relation will be just the inertial gravity waves.**

- **Rossby waves are low-frequency waves. Their periods are $T > 30$ days. That is, if the forcings are at periods shorter than a month, Rossby waves will not be excited.**

- **Their motion satisfies "Quasi-geostrophy".** Quasi-geostrophy is a complicated concept and has precise definition. But we will not cover it here. However, we will show the quasigeostrophic motion of Rossby waves in Figure 15.

4.8 Effects of side boundaries: coastally trapped waves

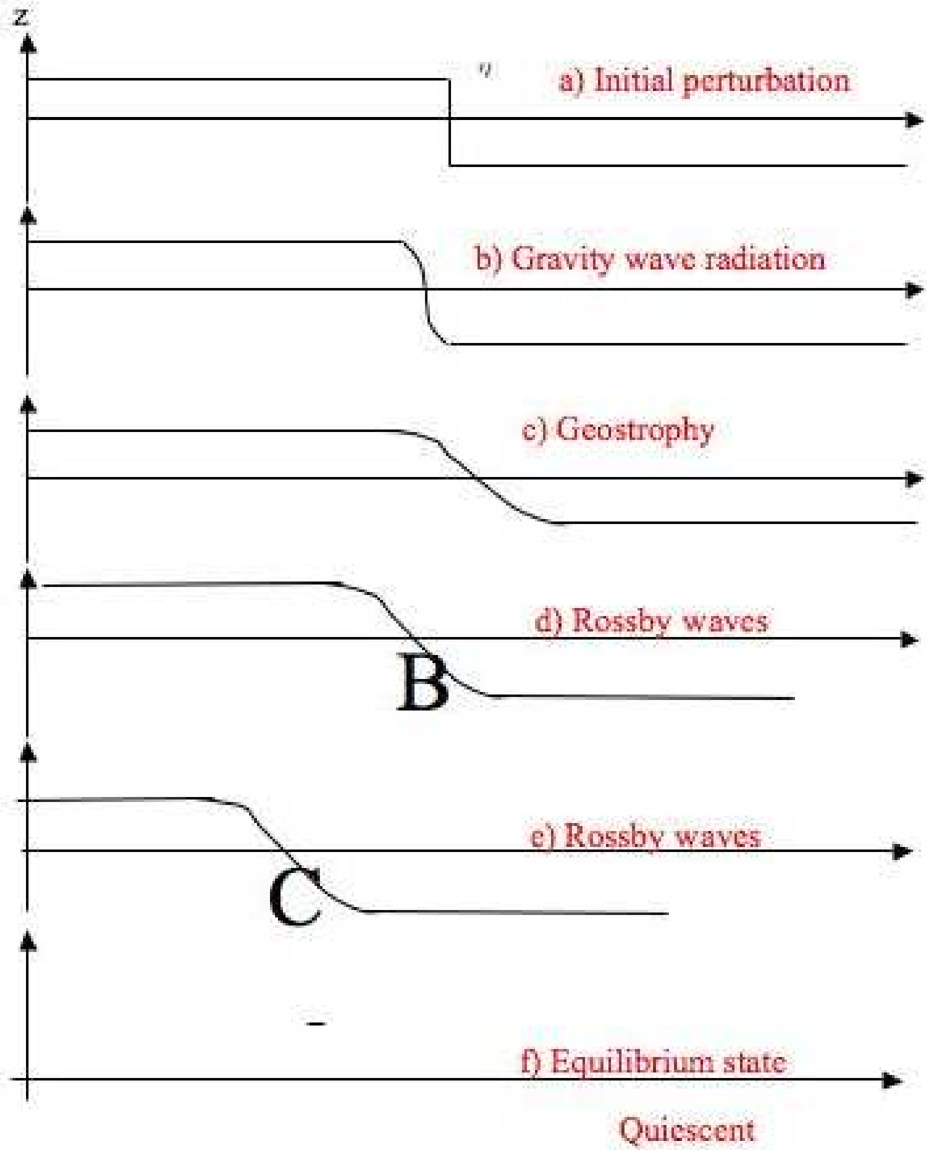

Figure 15: Schematic diagram showing Rossby wave propagation.

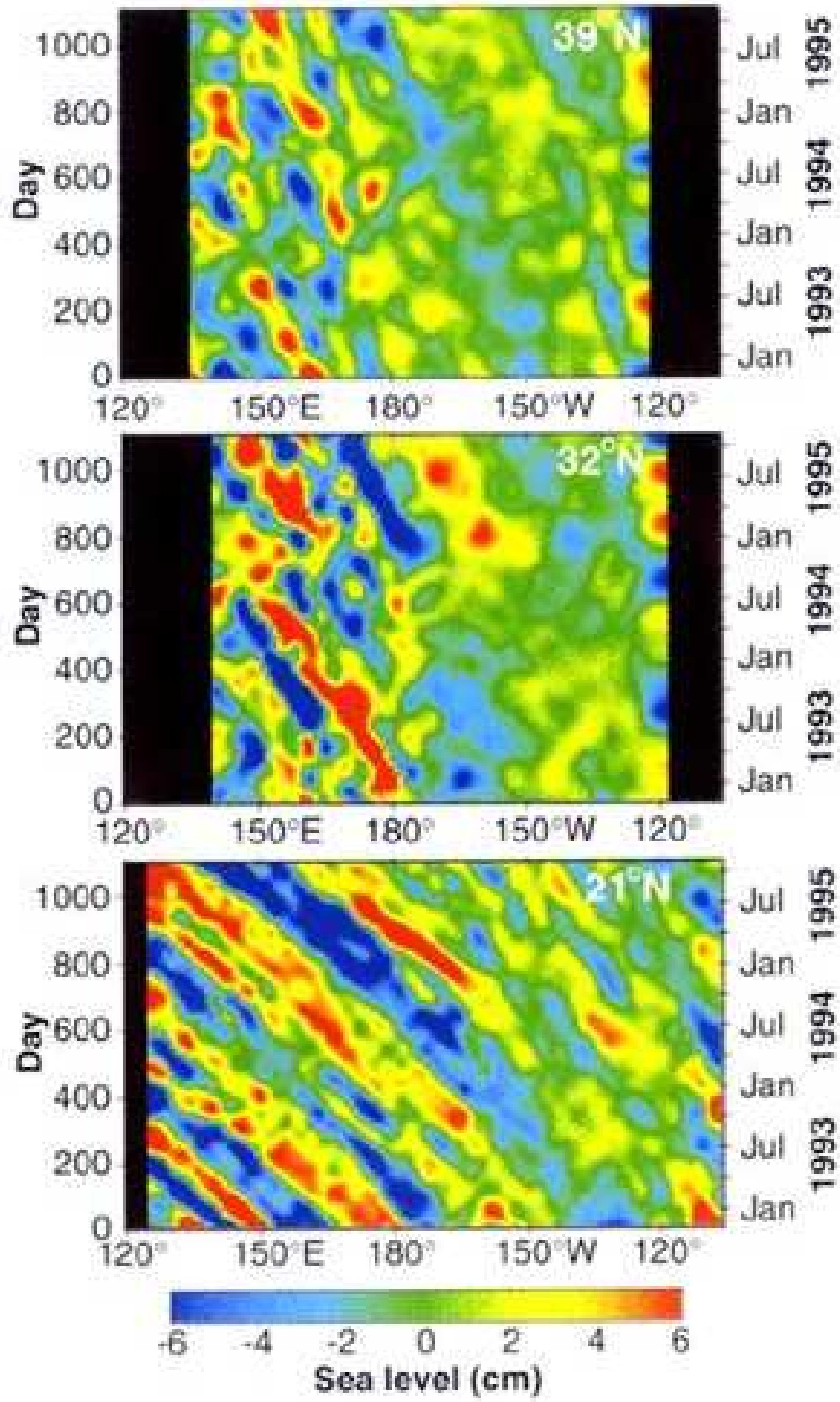

Figure 16: Longitude-time plot of Sea Surface Height (SSH) anomaly: Observed mid-latitude Rossby waves in the Pacific.

The presence of lateral boundary due to the continental margins distinguishes the ocean circulation from the atmospheric circulation. As we will see, the coastal ocean acts as a waveguide. Information can be transported rapidly from low latitudes to high latitudes. In Figure 17, we can see SSH anomalies hug the eastern Pacific coasts in the off equatorial latitudes.

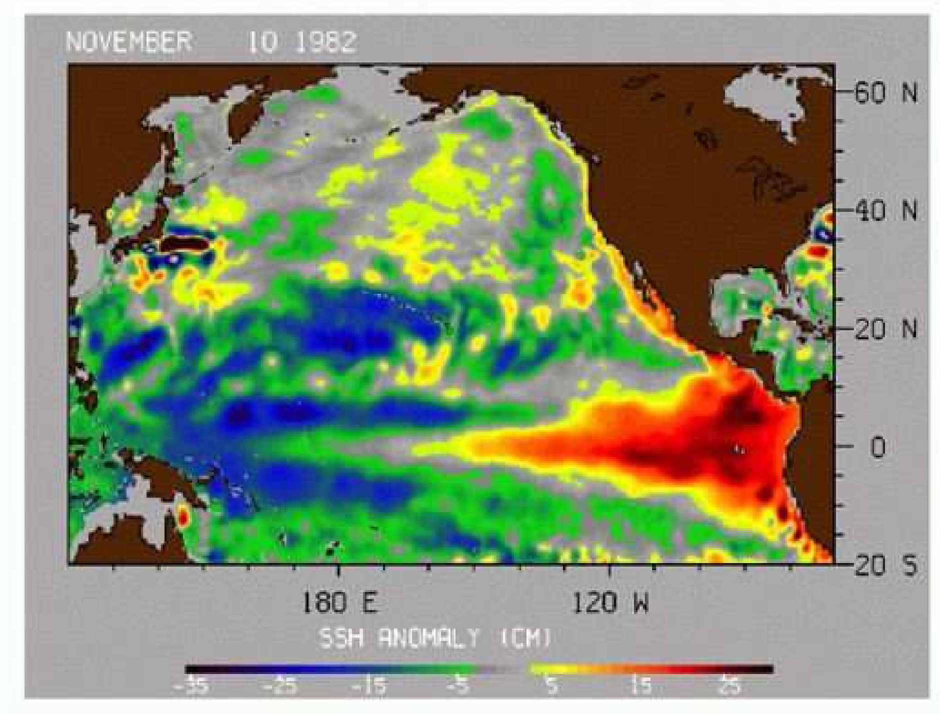

Figure 17: Observed SSHA.

a) Coastal Kelvin wave.

To simplify things, we look at the shallow water equation with $f = f_0 = const$, so Rossby waves do not exist in the system.

Boundary conditions are $y = 0, v = 0$. We find solutions with $v = 0$ everywhere. The equations become:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}, \tag{26a}$$

$$fu = -g\frac{\partial \eta}{\partial y}, \tag{26b}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0. \tag{26c}$$

By assuming a waveform solution you can easily derive the dispersion relation as:

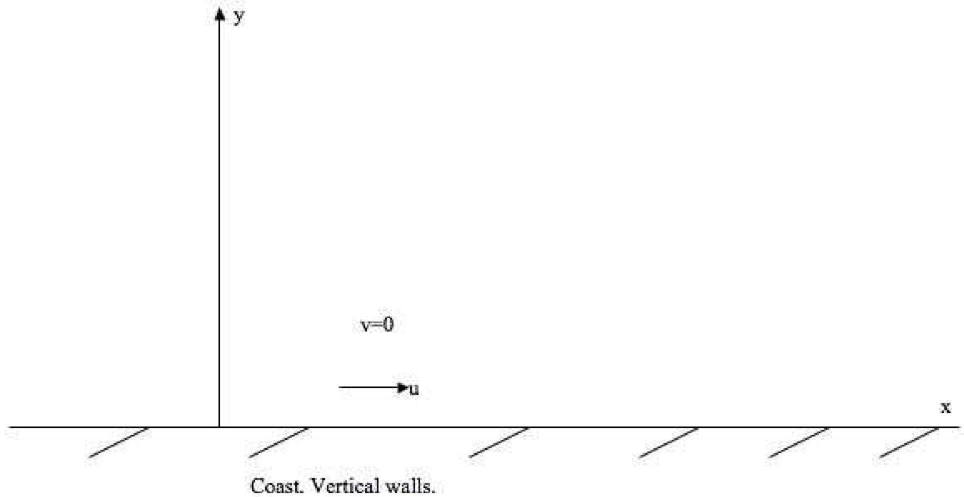

Figure 18: Schematic diagram showing the coastal vertical wall.

$\omega^2 = k^2c^2$, where $c = \sqrt{gH}$, or
$\omega = \pm kc$.
This is the same dispersion relation for "gravity waves in a non-rotating" system.

I will not derive the solution in detail here, but I'll write it down.
The solution for η is composed of two pieces F and G:
$F = Ae^{\frac{fy}{c}}F'(x+ct)$,
$G = Be^{\frac{-fy}{c}}G'(x-ct)$,
where both F' and G' are arbitrary functions. That is, one wave propagates westward (F) and the other one propagates eastward (G). [Determine wave propagation direction: Find constant phase line. $\Phi = x + ct = const = 0$, and $\Phi = x - ct = const = 0$. As time goes by ($t$ increases), x increases in $+x$ direction for G but in $-x$ direction for F.]

As y increases to infinity, F becomes infinity. For a perturbation that is initiated near the coast, amplitudes of the waves increase to infinity away from the coast. This is against the energy conservation and thus this piece of solution is impossible in real ocean.

As y increases to infinity, $G = 0$ and thus is the reasonable solution.

Therefore, the solution is:

$$G = Be^{\frac{-fy}{c}}G'(x-ct). \tag{27}$$

That is, only the gravity waves that propagate eastward (or with the coast to its right in the northern hemisphere) are the reasonable solutions for the coastal waves. They are referred to as "coastal Kelvin waves".

The dispersion relation for the coastal Kelvin waves are:

$$\omega = kc, \tag{28}$$

since only the one propagates with the coast to its right exists in NH in the system.

- **Coastal Kelvin waves dispersion character is the same as those long gravity waves in a non-rotating fluid. They are non-dispersive and they propagate with the coast to their right (left) in the NH (SH). Their structures, however, are trapped to the coast with an e-folding scale of $\frac{c}{f}$, which is the Rossby radius of deformation.** This is the reason for them to be called "coastal Kelvin wave" because they are trapped to the coast.

- *Coastal Kelvin waves can be generated by surface wind forcing close to the shore, tidal forces, reflection of other waves incident on a coast.*

In Figure 17, the SSH anomalies that hug the costs in subtropical-mid latitudes are coastal Kelvin waves generated by the incident equatorial Kelvin wave signals.

b). Continental shelf waves.

When the depth of the ocean at the coast varies, the Kelvin waves are modified, and a new class of waves arise. These waves are called: Continental shelf waves, or topographic Rossby waves.

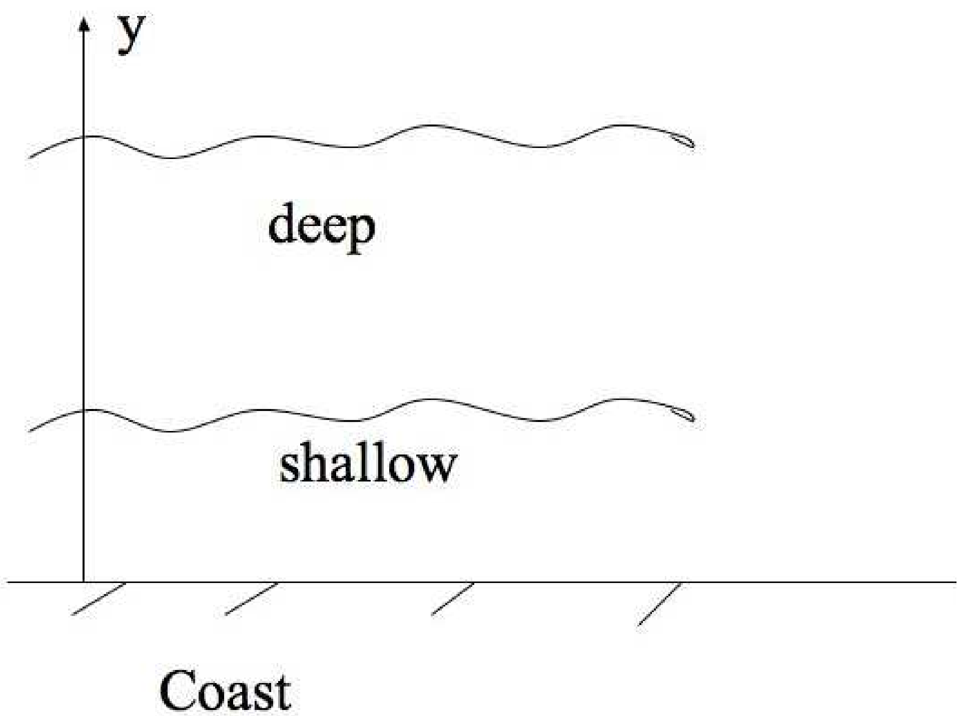

Figure 19a:

In this case, the shallow water equations on a f-plane are:

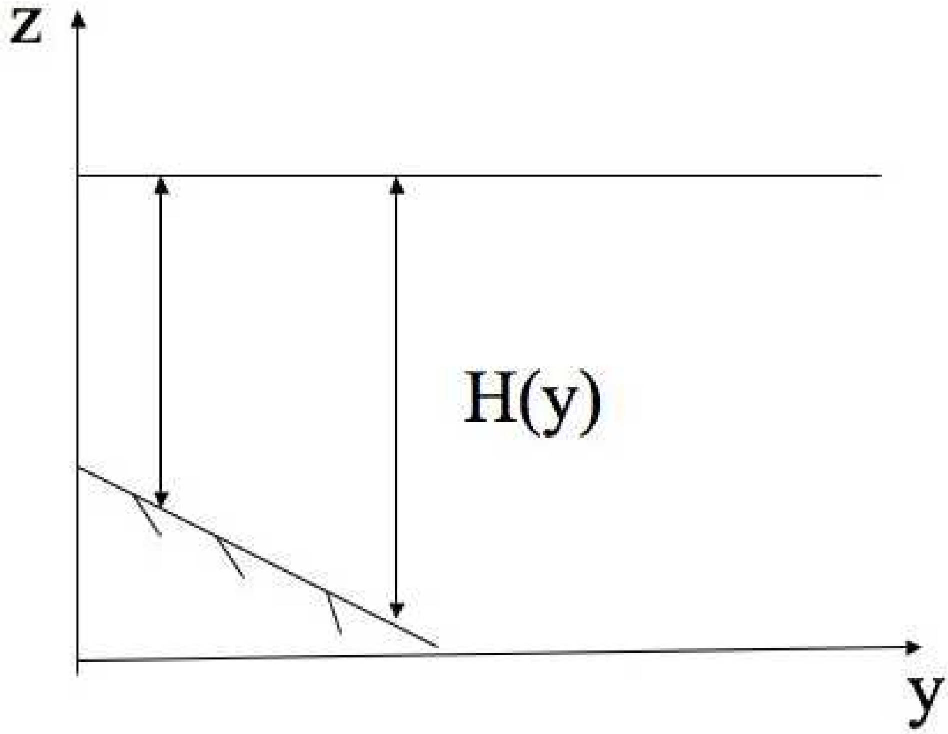

Figure 19b: Schematic diagram for shelf topography.

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}, \tag{29a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{29b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0. \tag{29c}$$

In this case $H = Hy$ and solution is more complicated to solve analytically. So we just provide some basic discussion regarding their energy dispersion.

Continental shelf waves propagate in the direction of coastal Kelvin waves, with the coast to its right (left) in NH (SH). A typical off-shore length scale is 30–60 km with $c{-}2m/s$. However, unlike Kelvin waves continental shelf waves are dispersive. The mechanism for its propagation can be demonstrated by potential vorticity conservation, which is $Q = \frac{\zeta+f}{H} = const$ and illustrated in Figure 20. [In an inviscid, unforced ocean, the potential vorticity is conserved for a moving water column.]

4.9 Equatorially trapped waves

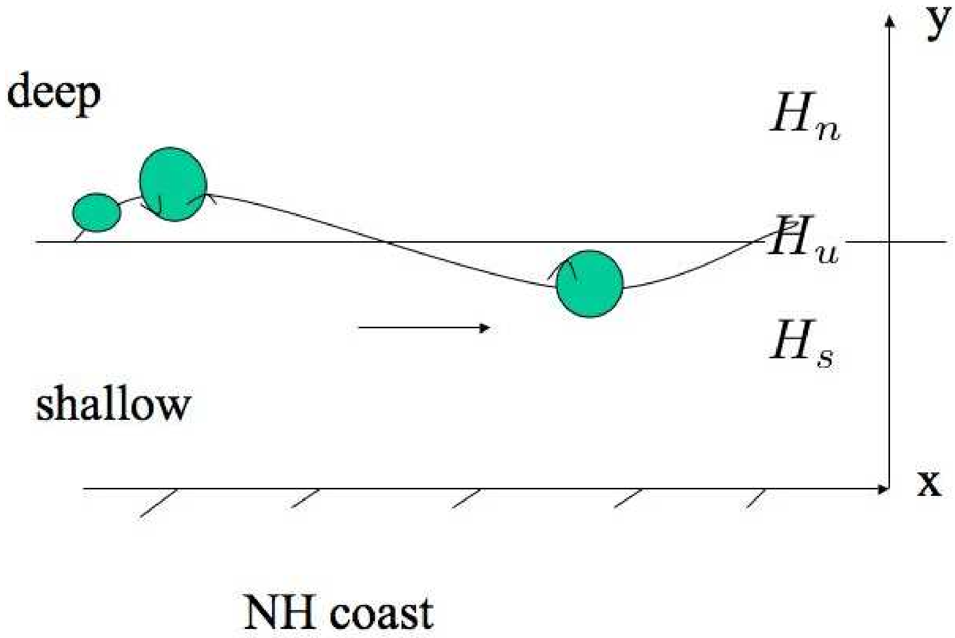

Figure 20: Shelf waves.

We noted earlier that a lateral boundary in the ocean can act as a waveguide, which guides the waves to propagate along it. These waves (coastal Kelvin waves and shelf waves) are "trapped" to the coast and thus perturbations associated with these waves decay away from the coast. For coast Kelvin waves, their amplitudes decay with an e-folding decay scale of Rossby radius $a = \frac{c}{f}$. In fact, *it is f that makes the coastal Kelvin waves trapped to the coast".*

As we shall see next, the equator acts as a waveguide. Waves are trapped to the equator because of "β, the variation of f with latitude", rather than "f". Because at the equator, $f = 2\Omega sin\phi = 0$. but $\beta \neq 0$.

Here in this class, we will just focus on discussing the physics without providing the mathematical details. For each type of wave, we will discuss the following aspects associated with the equatorially trapped waves:

- energy dispersion, phase speed (c_p) and group velocity (c_g);
- Solution character;
- Symmetric property;
- Forcing.

4.9.1. The equatorial Kelvin wave

a. Dispersion relation. **The dispersion relation for the equatorial Kelvin wave is:**

$$\omega = kc, \tag{30}$$

where c is a barotropic or baroclinic mode speed. If the ocean is barotropic, $c = c_0 = \sqrt{gH} = 200m/s$ for $H = 4000m$. Very fast. For a continuously stratified ocean, we have one barotropic mode and infinite number of baroclinic modes. Usually, the first baroclinic mode has a speed of $c_1 = 2 - 3m/s$.

As we can see, the equatorial Kelvin wave has the same dispersion relation as the coastal Kelvin wave and as the gravity waves in a non-rotating fluid ($f = 0$). They are non-dispersive since $c_g = \frac{\partial \omega}{\partial k} = c$ is independent of frequency. $c_g = c_p$. Both phase and group velocity (energy) propagate eastward.

Equatorial Kelvin waves propagate eastward.

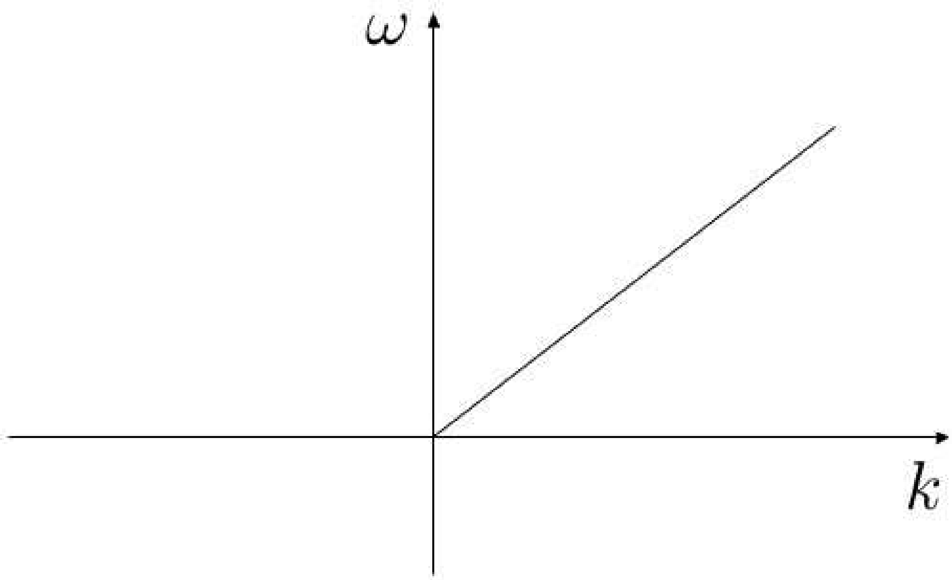

Figure 21: Dispersion relation curve for Kelvin wave.

b. Solution. **The solutions to Kelvin wave are:**

$\eta = e^{-\frac{\beta y^2}{2c}} G'(x - ct),$
$u = \frac{g}{c} e^{-\frac{\beta y^2}{2c}} G'(x - ct),$
$v = 0.$

As we can see from the above solutions, Kelvin waves propagate eastward with a speed of c. Their amplitudes are the largest at the equator and decay poleward to each side of the equator with an e-folding scale of $y = \sqrt{\frac{2c}{\beta}}$. They are like two coastal Kelvin waves propagating back-to-back along the equator

(where $f = 0$), one is to the north of the equator and the other to the south. In other words, the equator behaves like an artificial coastline for each hemisphere.

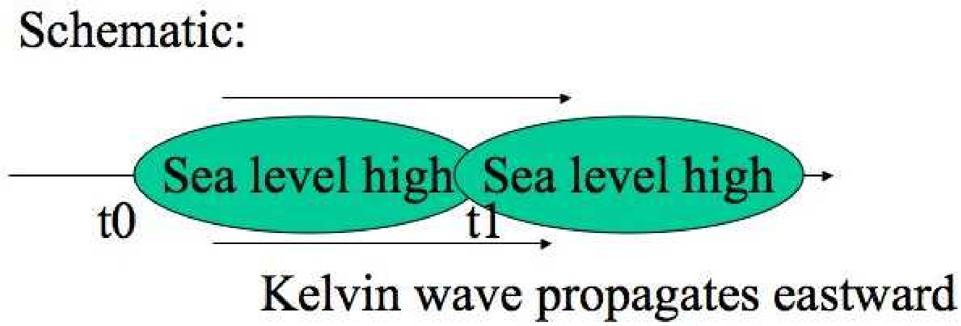

Figure 22: Schematic for Kelvin waves.

Often, a naturally occurring meridional length scale in the tropics is:
$a_e = \sqrt{\frac{c}{2\beta}}$ (see Gill 11).

It is called the "equatorial radius of deformation", and is analogous to the Rossby radius of deformation at off-equatorial latitudes.

Again, the equatorially trapped nature of waves results from the presence of β. If we set $\beta = 0$ in the above solutions then the waves are not trapped anymore! Recall that the trapping of coastal Kelvin waves is due to the presence of "f".

For the first baroclinic mode with $c_1 - 3m/s$, $a_1 - 250km$.

(visit http://www.pmel.noaa.gov/tao/vis/gif/t-dyn-med.gif, which shows real time temperature and dynamic height associated with Kelvin waves pauses).

c. Symmetric property. As we see from the solutions, Kelvin waves are symmetric about the equator. They are symmetric waves. Both u and η are symmetric about the equator. $v = 0$.

d. Forcing. At the equator, the Kelvin waves are mostly generated by the changing winds. That is, variations of winds with time.

Assume the ocean is calm initially. Then all of a sudden westerly winds (winds blowing from west to east) start to blow along the equator. You will see Kelvin wave pause associated with a high sea level and a deepened thermocline in a simple 2-layer system. This is because westerly winds will cause eastward currents initially, and Coriolis force off the equator tend to pull the motion toward right (left) in the NH (SH). As a result, there is water converges on the equator and thus pushes the thermocline downward and raise the sea level. *Amplitudes of Kelvin waves are proportional to the "zonal integral" of the westerly winds (derivation is not provided).*

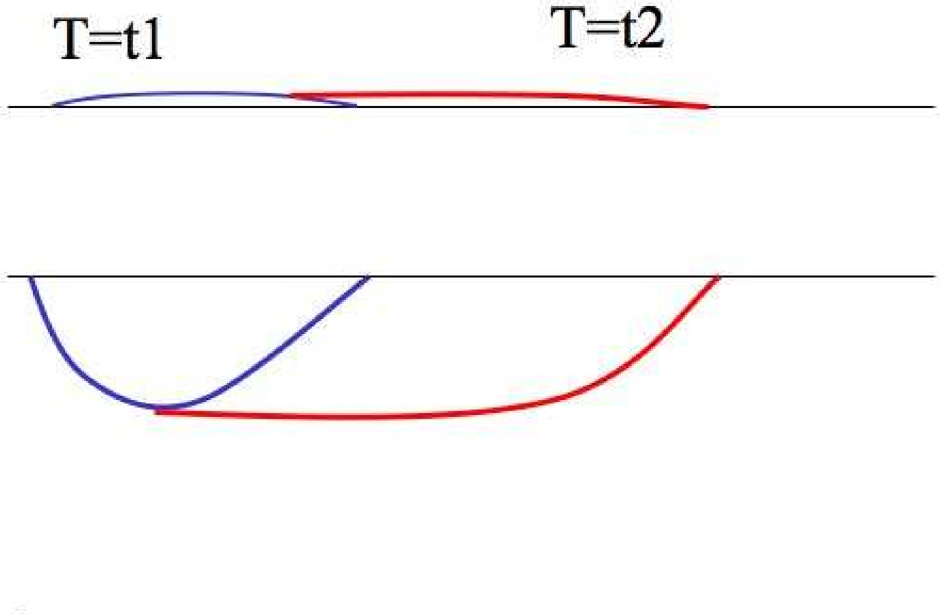

Figure 23: Schematic diagram showing equatorial Kelvin waves forced by westerly winds.

Forced Kelvin and Rossby waves from a linear model. Gill 11.11 (cited from McCreary 1978). Coastal Kelvin waves are clearly seen along the eastern ocean boundary after the equatorial Kelvin waves arrive at the boundary. At first, the coastal Kelvin waves are trapped to the coast because they propagate fast (at speed of c). Then as time goes by, they become entrapped due to the westward radiation of the Rossby waves. We can see from the figure that speeds of Rossby waves decrease with latitude increase.

4.9.2. The equatorial Rossby wave.

a. Dispersion relation.

$\omega = -\frac{\beta k}{k^2+\frac{\beta}{c}(2n+1)}$,

where $n = 1, 2...$ **is the order number for Hermite function (or Parabolic cylinder function).**

For long Rossby waves, k^2 **is small and thus the dispersion can be written as:** $\omega = -\frac{ck}{2n+1}$.

$C_g = -\frac{c}{2n+1}$ **is independent of** ω**, and thus long Rossby waves are non-dispersive, and** $c_g = c_p = const$. **Therefore, both phase and group velocity propagate westward, as the long Rossby waves in mid-latitude.**

For $c_1 = 3m/s$, $n = 1$, **and** $k = \frac{2\pi}{1000km}$, $T = \frac{2\pi}{\omega} = 35 days$, **which is also consistent with the highest frequency (shortest period) for baroclinic Rossby waves.**

For short Rossby waves, k^2 **is larger and thus,**

$\omega = -\frac{\beta}{k}$. The same as dispersion relation as the short Rossby waves at mid-latitude Rossby waves. They are dispersive waves.

$c_g = \frac{\beta}{k^2}$ and $c_p = -\frac{\beta}{k^2}$.

Short Rossby waves group velocity is eastward, but phase speed is westward. They are two short to be efficiently excited in the ocean interior. Mixing also acts strongly on these short waves. As discussed in the mid-latitude Rossby waves section, these waves exist only near the western boundary. Their energy is killed by mixing and thus forming the WBC.

b. Solution. The solutions for equatorial Rossby wave involve Hermite function. That is, the solutions are expanded in y direction by Hermite function.

$v_n = D_n(\sqrt{\frac{2\beta}{c}}y)cos(kx-\omega t) = 2^{-n/2}H_n(\sqrt{\frac{\beta}{c}}y)e^{-\frac{\beta y^2}{2c}}cos(kx-\omega t)$,

$\eta = \frac{c}{2g}\sqrt{2\beta c}[\frac{D_{n+1}(\sqrt{\frac{2\beta}{c}}y)}{kc-\omega} + \frac{nD_{n-1}(\sqrt{\frac{2\beta}{c}}y)}{kc+\omega}]sin(kx-\omega t)$,

$u_n = \sqrt{\frac{\beta c}{2}}[\frac{D_{n+1}(\sqrt{\frac{2\beta}{c}}y)}{kc-\omega} - \frac{nD_{n-1}(\sqrt{\frac{2\beta}{c}}y)}{kc+\omega}]sin(kx-\omega t)$,

where D_n is parabolic cylinder function of order n, and H_n is a Hermite polynomial of order n.

For even numbers of "n", D_n is symmetric; For odd numbers of "n", D_n is antisymmetric.

From these solutions, we can see that *amplitudes of Rossby waves decay poleward and oscillatory in y.* They are equatorially trapped: with an e-folding scale of $a_e = \sqrt{\frac{c}{2\beta}}$.

c. Symmetric property. We use symmetric properties of u or p (or η) to define symmetric properties of waves. That is, if u or p (or η) are symmetric (antisymmetric) about the equator, then we referred to these waves as symmetric (antisymmetric) waves.

From the above solutions, if v_n is antisymmetric, u_n and η_n are symmetric. Rossby waves can be both symmetric or antisymmetric, depending on the forcing structures. If the forcing is symmetric, symmetric Rossby waves will be excited. If the forcing is antisymmetric, then antisymmetric Rossby waves will be generated.

d. Forcing. Long equatorial Rossby waves can be efficiently excited by the change of large spatial scale winds. See Figure 24. Since westerly winds cause equatorial convergence due to Coriolis effects, in the off equatorial region for both NH and SH, divergence is generated due to water flows toward the equator. Therefore, an equatorial deepened thermocline (which represents high sea level for baroclinic modes) occurs, and off equatorial shallowed thermocline is produced. Since the forcing winds are symmetric about the equator, only symmetric Rossby waves are generated.

BOUNDARY REFLECTION FOR ROSSBY AND KELVIN WAVES

Physics: Kelvin waves propagate eastward, both for phase and group velocity. After Kelvin waves impinge onto the eastern ocean boundary, part of the energy is reflected back into the ocean interior as "long Rossby waves", since the long Rossby waves' energy propagate westward and thus can take the energy away from the eastern boundary; part of the energy propagates poleward via coastal Kelvin waves. Therefore, variability that initiates at the equator can propagate to the mid-latitude quickly through coastal Kelvin waves.

Long Rossby waves propagate westward. As long Rossby waves impinge onto the western ocean boundary, part of the energy is reflected back into the ocean interior as Kelvin waves, and part of it as short Rossby waves, both of which have eastward group velocity and thus can take the energy away from the western ocean boundary. Since short Rossby waves are short and slow, their energy is primarily killed by mixing near the western boundary.

Now we have a question: why don't we have coastal Kelvin waves that take the energy poleward near the western boundary? [Prompt the class]

4.9.3. Equatorially trapped inertial gravity waves

a. Dispersion relation.

$\omega^2 = k^2c^2 + \beta c(2n+1)$,

where $n = 1, 2...$ is the order number for Hermite function (or Parabolic cylinder function).

These waves are called inertial gravity waves, which is similar to the inertial gravity waves at mid-latitude. They are gravity waves under the influence of β (f) at the equator (mid-latitude). *They are dispersive waves, and their energy and phase can propagate both eastward and westward.*

Their minimum frequency is: $\omega_{min} = \sqrt{\beta c(2n+1)}$. For $n = 1$, $c_1 = 3m/s$, and $k = \frac{2\pi}{1000km}$,

$\omega_{min} = 2\pi \times 3.8 day^{-1}$,

and thus the longest period is 3.8 day. So gravity waves periods are shorter than a few days.

b. Solution. **The solutions for equatorially trapped inertial gravity waves have the same form as the solutions for Rossby waves.** *They are oscillatory in y direction and at the same time decay poleward.* **They are equatorially trapped: with an e-folding scale of $a_e = \sqrt{\frac{c}{2\beta}}$.**

c. Symmetric property. We use symmetric properties of u or p (or η) to define symmetric properties of waves. That is, if u or p (or η) are symmetric (antisymmetric) about the equator, then we referred to these waves as symmetric (antisymmetric) waves. Inertial gravity waves can be both symmetric or antisymmetric, depending on the forcing structures. If the forcing is symmetric, symmetric gravity waves will be excited. If the forcing is antisymmetric, then antisymmetric gravity waves will be generated.

d. Forcing. Winds associated with synoptic weather phenomena, or gusty winds with periods of a few hours.

4.9.4. The mixed Rossby gravity wave: Yanai wave

a. Dispersion relation.
$k = -\frac{\beta}{\omega} + \frac{\omega}{c}$.

This wave is interesting because (i) as frequency is high (ω is large), the dispersion becomes $k = \frac{\omega}{c}$, which is Kelvin wave's dispersion relation; (ii) as frequency is low, the dispersion becomes $k = -\frac{\beta}{\omega}$, which is short Rossby wave's dispersion relation. Therefore, energy dispersion of this type of wave behaves like short Rossby wave at low frequency, and like Kelvin wave (whose restoring force is gravity) at high frequency. So they are called "mixed Rossby gravity waves", or Yanai waves since Yanai first discovered it.

So, at low frequency, Phase speed of Yanai waves is westward but group velocity is eastward. At high frequency, phase speed of Yanai waves is eastward and group velocity is also eastward. *Group velocity of Yanai waves is eastward. Phase speed can be eastward or westward, depending on the frequency.* **They are dispersive waves. But at high frequency they are close to be non-dispersive.**

See Figure 30 for Yanai wave dispersion curve on dispersion relation diagram.

b. Solution. **The solutions for Yanai wave are the $n = 0$ case for the solutions of Rossby waves. Therefore,**

$v_0 = D_0(\sqrt{\frac{2\beta}{c}}y)cos(kx - \omega t)$.

c. Symmetric property. As we can see, v is symmetric for Yanai waves and thus antisymmetric for u and p. Yanai waves are antisymmetric waves; their zonal velocity u and pressure p are antisymmetric about the equator, and meridional velocity v is symmetric about the equator.

d. Forcing. Oscillatory meridional winds or anti-symmetric part of zonal winds.

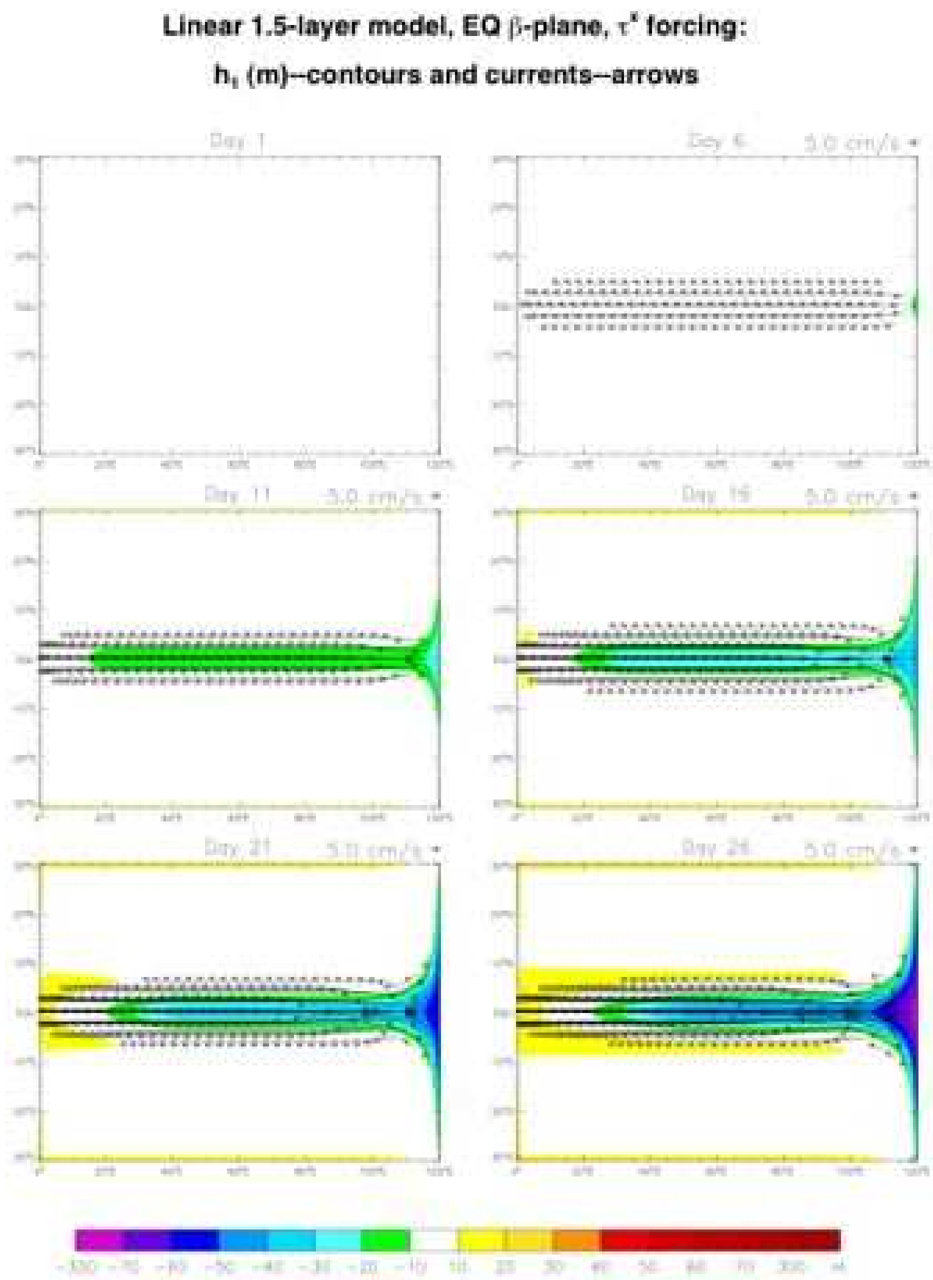

Figure 24: Wind-driven equaotiral Kelvin and Rossby waves from a $1\frac{1}{2}$-layer model.

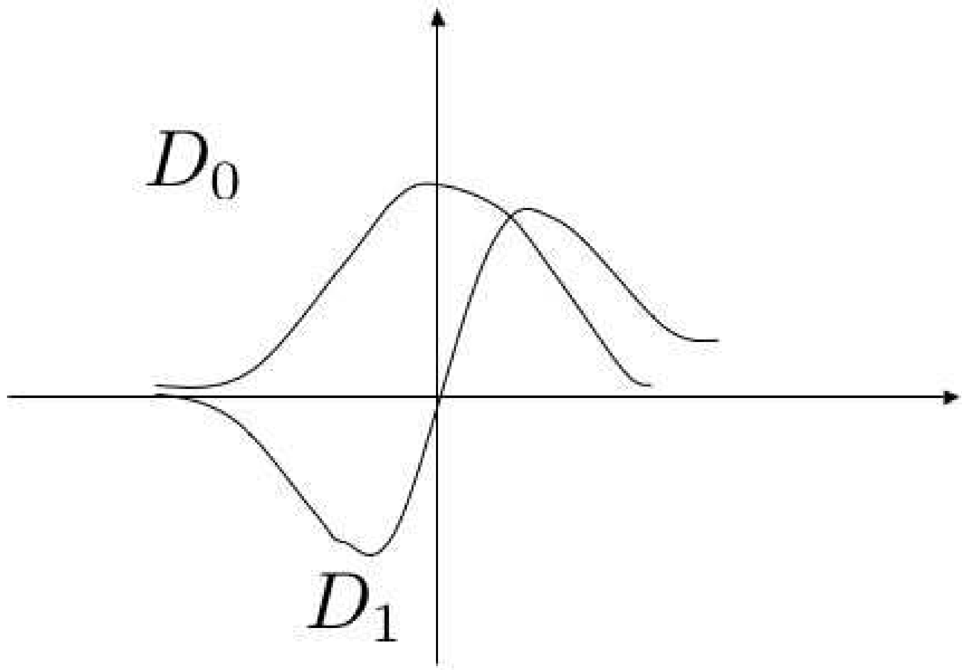

Figure 25: Schematic diagram for the first a few D_n.

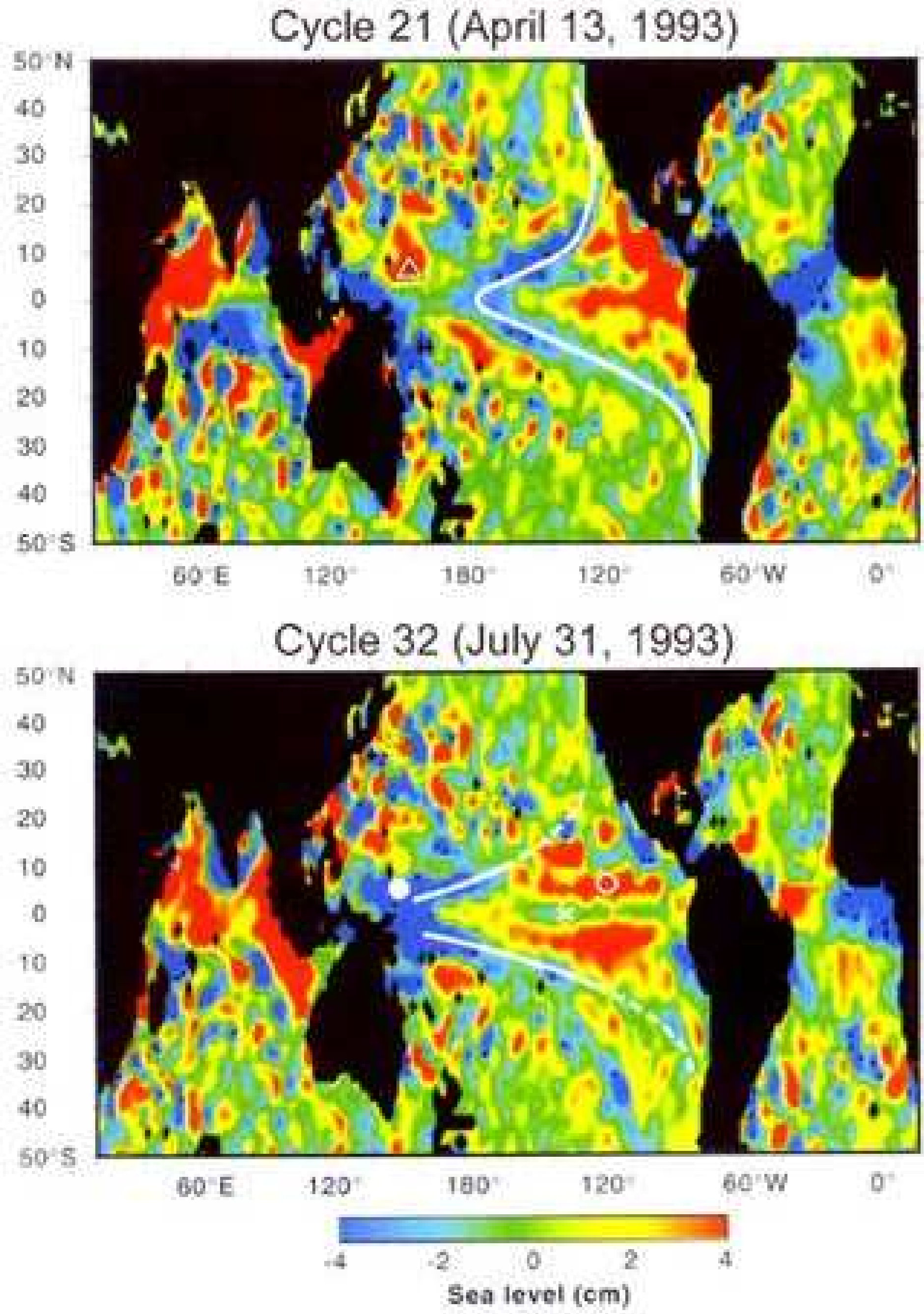

Figure 26: Observed equtorial Rossby waves from satellite SSH anomaly.

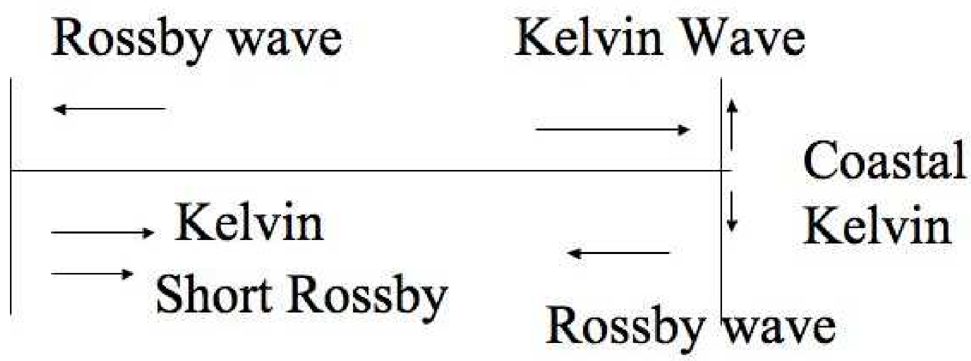

Figure 27: Schematic diagrams for equatorial Kelvin and Rossby waves boundary reflection.

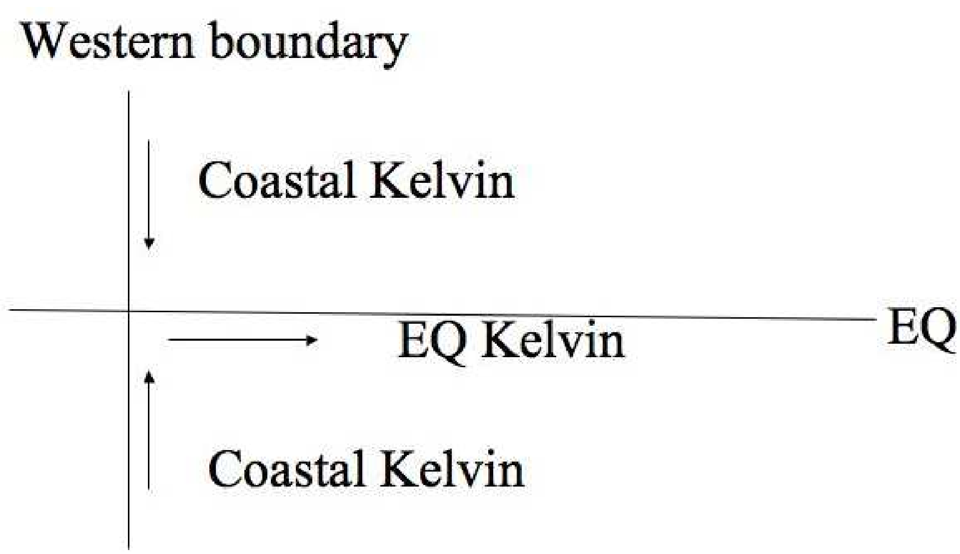

Figure 28: Schematic diagram for coastal Kelvin waves.

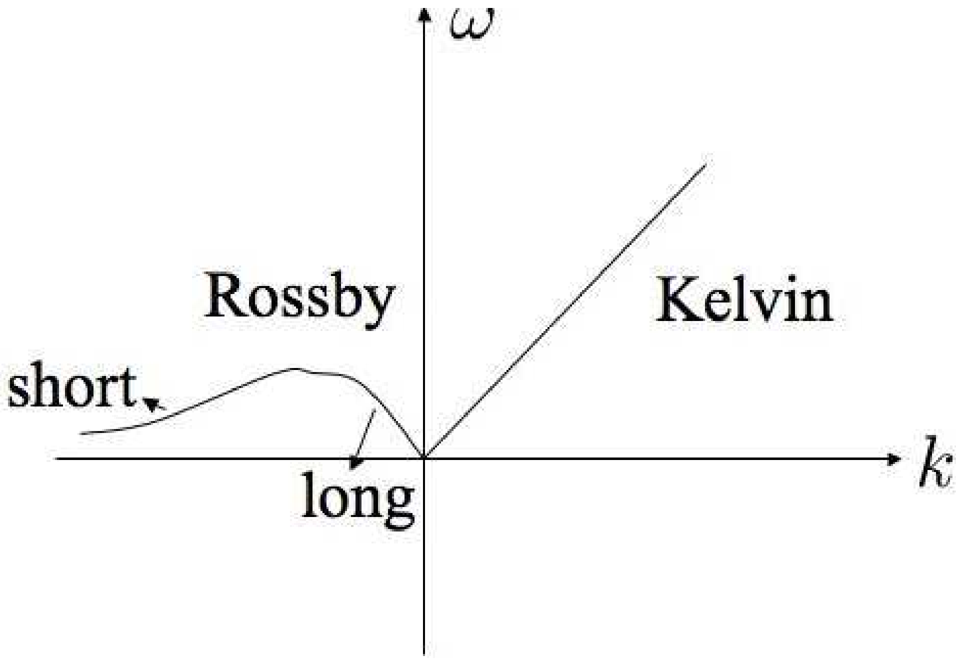

Figure 29: Dispersion relations for equatorial Kelvin and Rossby waves.

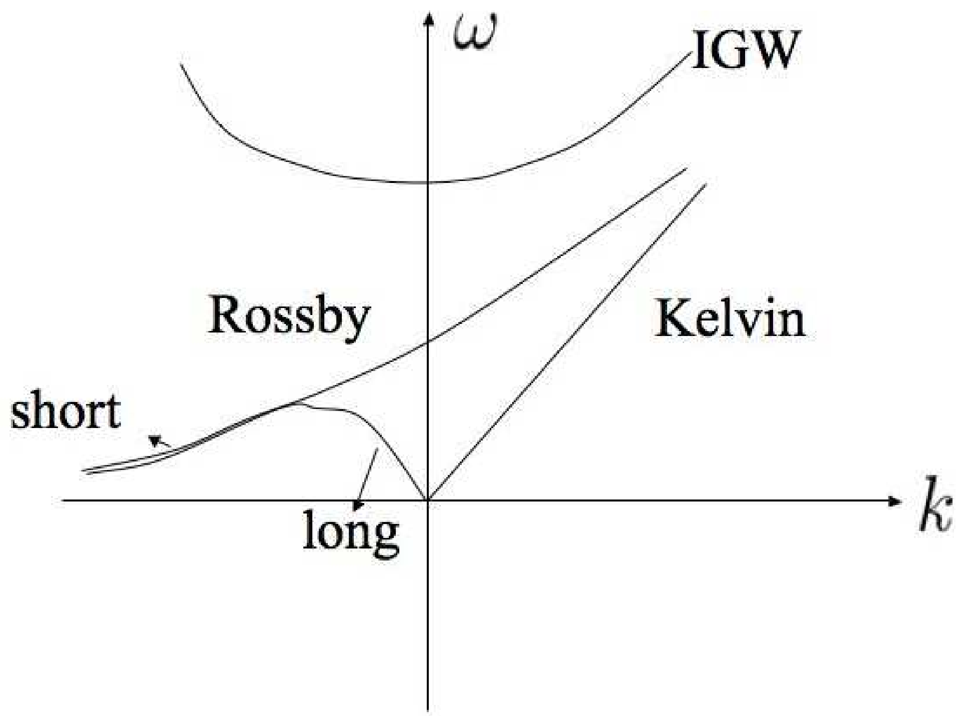

Figure 30: Yanai wave dispersion curve on dispersion relation diagram.

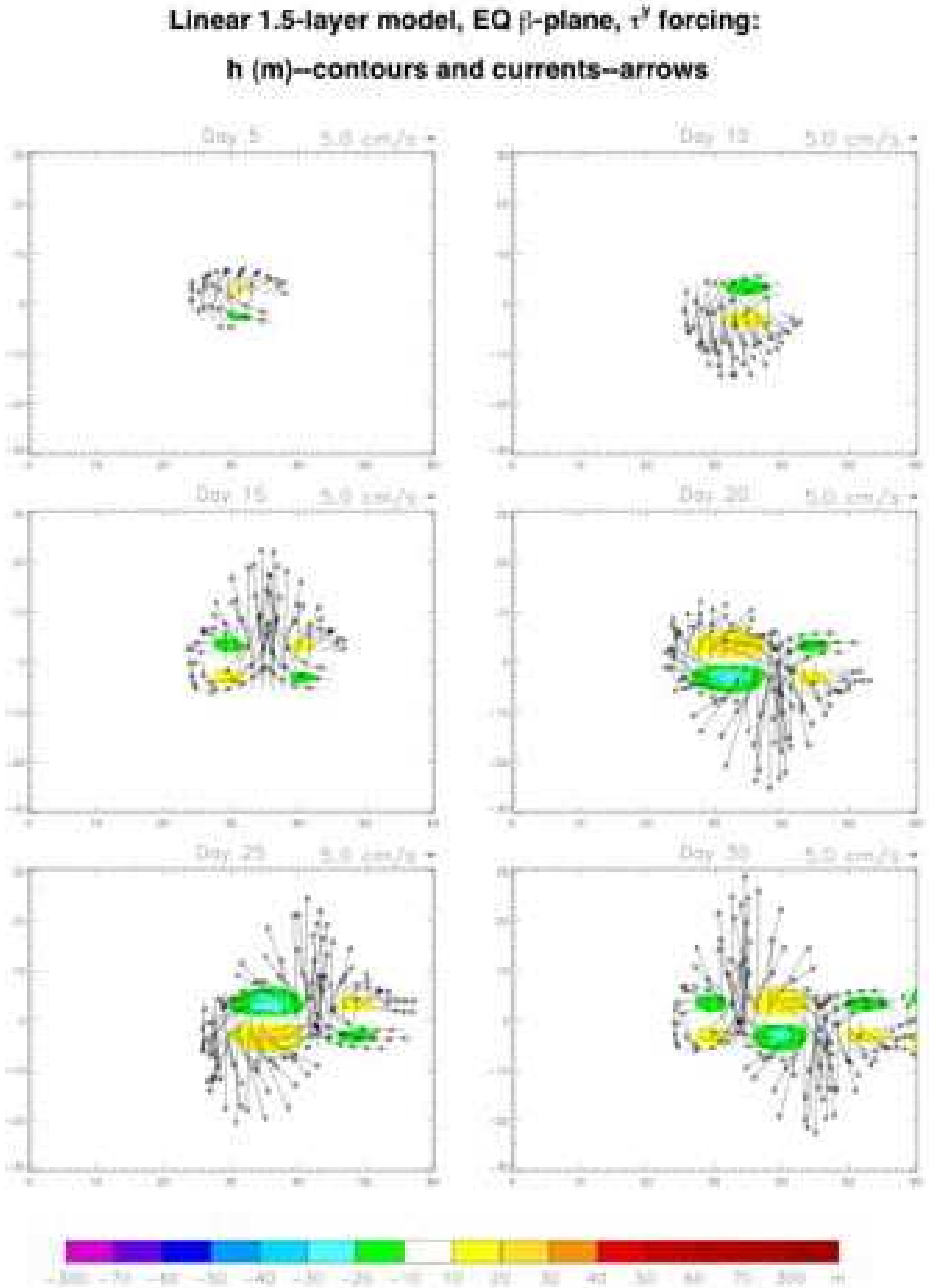

Figure 31: Structures for Yanai waves.