Polarization

Poynting Vector: specifes magnitude and direction of energy transport by an E-M Field:

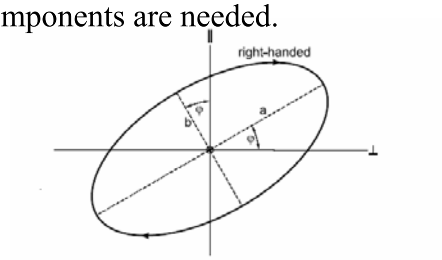
$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

E and **H** are perpendicular to direction of energy transport.

It is customary to specify state of polarization in terms of E fields but magnetic fields do just as well. Only assumption: **E** is time harmonic: $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})\exp(-i\omega t)$

To specify the polarization state of **E** we need the real field.
Because E lies in a plane perpendicular to S only two components are needed.

$$E_{\perp} = a_{\perp}\exp\{-i(\theta_{\perp}+\omega t)\}$$
$$E_{||} = \mathrm{a}_{||}\exp\{(-i(\theta_{||}+\omega t)\}$$



These equations describe an arbitrarily oriented ellipse (traced out by the tip of the E-field vector) of arbitrary ellipticity defined as the ratio of the minor to major axis lengths.

When the two phases are equal curve is straight line with slope equal to ratio of amplitudes.

When phases differ by $\pi/2$ the curve traced out is an ellipse with principle axes aligned along coordinate axes straight line and with lengths of semi-axes equal to amplitudes. If amplitudes are equal a circle results.

Ellipsometric parameters:

- Amplitudes
- Phases
- Azimuth: angle between the major axis and a reference axis
- Handedness: the rotational sense in which it is traced out in time.
- Completely polarized light: there is *complete* correlation between two orthogonal components
- Partially polarized light: *partial* correlation between the two orthogonal components
- Unpolarized light results when there is *no* correlation

Define/determine Stokes (ellipsometric) parameters:

- All that we can measure is time-averaged irradiances.
  - Cannot watch the tip of an electric vector rotating at 10^{15} Hz!
  - Can measure $a_{\perp}$, $a_{||}$, $\theta_{\perp}$, $\theta_{||}$

Experiment 1:

- Measure time-averaged irradiance of a beam: $I = \langle S \rangle = E_{\parallel} E_{\parallel}^{*} + E_{\perp} E_{\perp}^{*} = a_{\parallel}^{2} + a_{\perp}^{2}$

Experiment 2:

- *Using ideal linear polarizer*, first, measure along $\mathbf{e}_{\parallel}$ and then along $\mathbf{e}_{\perp}$ and take difference*:* $Q = E_{\parallel} E_{\parallel}^{*} - E_{\perp} E_{\perp}^{*}$

Can compute amplitudes: $a_{\parallel} = 1/2\,(I + Q)$ and $a_{\perp} = 1/2\,(I - Q)$

Experiment 3:

How about phases?

- must transmit a bit of both orthogonal components of an electric field.
- align a linear polarizing filter with its transmission axis at 45° to $\mathbf{e}_{\parallel}$: transmitted amplitude is $\frac{1}{\sqrt{2}}(E_{\parallel} + E_{\perp})$
- Rotate 90°: $\frac{1}{\sqrt{2}}(E_{\parallel} - E_{\perp})$

Difference in irradiances: $U = E_{\parallel} E_{\perp}^{*} + E_{\perp} E_{\parallel}^{*} = 2 a_{\parallel} a_{\perp} \cos(\delta)$ where $\delta = \theta_{\parallel} - \theta_{\perp}$

Measurement of *I, Q*, and *U* is sufficient to obtain cos(δ) but not handedness:

cos (δ) = cos(-δ)

Experiment 4 (requires set of ideal circular polarizers):

- introduce new set of complex basis vectors:

$$\mathbf{e}_{\mathrm{R}} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\parallel} + i\mathbf{e}_{\perp}),\ \mathbf{e}_{\mathrm{L}} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\parallel} - i\mathbf{e}_{\perp})$$

$$E = E_{\mathrm{R}}\mathbf{e}_{\mathrm{R}} + E_{\mathrm{L}}\mathbf{e}_{\mathrm{L}};\ E_{\mathrm{R}} = \frac{1}{\sqrt{2}}(E_{\parallel} - iE_{\perp}),\ E_{\mathrm{L}} = \frac{1}{\sqrt{2}}(E_{\parallel} + iE_{\perp})$$

- *Using ideal right- circular polarizer* measure transmitted irradiance $E_{\mathrm{R}} E_{\mathrm{R}}^{*}$;
  Using ideal left-circular polarizer measure transmitted irradiance $E_{\mathrm{L}} E_{\mathrm{L}}^{*}$

  $V = E_{\mathrm{R}} E_{\mathrm{R}}^{*} - E_{\mathrm{L}} E_{\mathrm{L}}^{*} = i(E_{\parallel} E_{\perp}^{*} - E_{\perp} E_{\parallel}^{*}) = 2\Im(E_{\parallel} E_{\perp}^{*}) = 2 a_{\parallel} a_{\perp} \sin(\delta)$

Now can compute sign of δ

***I, Q,* and *U and V are Stokes parameters:* $I^2 = Q^2 + U^2 + V^2$** for fully polarized light.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I_u \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{lp} \\ Q \\ U \\ 0 \end{pmatrix} + \begin{pmatrix} I_{cp} \\ 0 \\ 0 \\ V \end{pmatrix}$$

Degree of elliptical polarization: $\frac{I_p}{I_p + I_u} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$

Degree of linear polarization: $\frac{I_{lp}}{I} = \frac{\sqrt{Q^2 + U^2}}{I}$

Degree of circular polarization: $\frac{I_{cp}}{I} = \frac{V}{I}$

Polarization upon scattering

Polarization of Scattered Sunlight

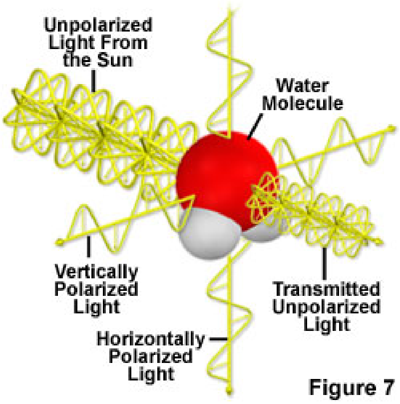
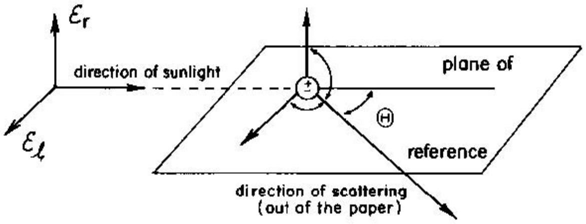
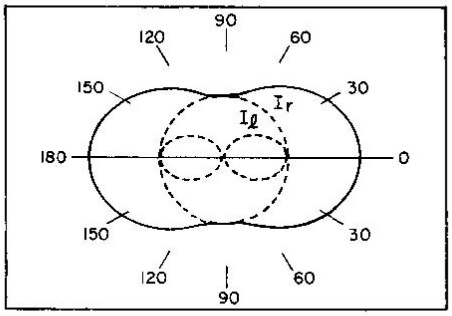


Figure 7



$$P(\Theta) = \frac{3}{4}(1 + \cos^2 \Theta)$$

(a)



Transformations of Stokes Parameters: The Mueller Matrix

- $\mathbf{I}_{\text{in}}$: Stokes parameters of an input beam; $\mathbf{I}_{\text{out}}$: Stokes parameters of an output beam
  $$\mathbf{I}_{\text{out}} = \mathbf{M}\mathbf{I}_{\text{in}}$$
  M is a 4 × 4 matrix called the *Mueller matrix* characteristic of the optical element
- For phase function with polarization:
  $$\mathbf{I}_{\text{out}} = c\mathbf{P}\mathbf{I}_{\text{in}}$$
  P is a 4 × 4 matrix called the *phase matrix* (where each component is a function of scattering angle) characteristic of the scatterer. c is normalizing solid angle:
  $$c = C_{\text{sca}}/(4\pi R^2)$$
- For randomly oriented particles: $\begin{pmatrix} I_{sca} \\ Q_{sca} \\ U_{sca} \\ V_{sca} \end{pmatrix} = c \begin{pmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & -P_{34} & P_{44} \end{pmatrix} \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{pmatrix}$

Back to Rayleigh's limit (x <<1):

Remember, E-field causes charge separation in particle. The induced dipole moment (charge distance) is: $\mathbf{p} = \alpha\mathbf{E} = \alpha\mathbf{E}_0 \exp(i\omega t)$, where α is polarizability.

Scattered field produced by accelerating charge is proportional to second derivative of **p**:
$\left|\mathbf{E}_s\right| \propto \left|\frac{\partial^2\mathbf{p}}{\partial t^2}\right| \propto \omega^2$ so scattered ***power*** is proportional to ω^4 (again, Raylieigh's law).
Write the two polarization components of the scattered field as:

$$E_{\|} = \frac{E_{\|,0}e^{-ik(R-ct)}k^2\alpha\cos(\Theta)}{R} \text{ and } E_{\perp} = \frac{E_{\perp,0}e^{-ik(R-ct)}k^2\alpha}{R}$$

R is distance from particle, k is wavenumber.

Scattered irradiance: $I_{\|} = I_{\|,0}\frac{(2\pi/\lambda)^4\alpha^2}{R^2}\cos^2(\Theta)$ and $I_{\perp} = I_{\perp,0}\frac{(2\pi/\lambda)^4\alpha^2}{R^2}$
For unpolarized incident radiation (such as sunlight): $I_{\|,0} = I_{\perp,0} = I_0/2$ so scattered irradiance: $I(\Theta) = I_0\frac{(2\pi/\lambda)^4\alpha^2}{R^2}\frac{1+\cos^2(\Theta)}{2}$ and $P(\Theta) = \frac{3}{4}(1+\cos^2(\Theta))$, Rayleigh phase function.

Mie Theory for general solution to scattering from homogeneous sphere:

Overview:
1) Express electric field inside and outside sphere in a vector spherical harmonic expansion, which satisfies Maxwell's equations.
2) Apply boundary conditions - match transverse fields at sphere surface to obtain outgoing spherical wave coefficients a_{n} and b_{n}.
3) Use series involving a_{n} and b_{n} to obtain extinction and scattering efficiencies (Q_{ext} and Q_{sca}).
4) Use series in Mie angular functions to obtain scattering amplitude functions $S_1(\Theta)$ and $S_2(\Theta)$, from which phase function is derived.

Scattering Amplitudes

Complex functions that describe pattern and polarization of scattered electric field in terms of the incident field. The far-electric field (far from sphere boundary: $R >> kr^2$):

$$\begin{pmatrix} E_{\|} \\ E_{\perp} \end{pmatrix}_{scat} = \frac{\exp(-ikR+ikz)}{ikR}\begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix}\begin{pmatrix} E_{\|,0} \\ E_{\perp,0} \end{pmatrix}_{inc} \text{ where } \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \text{ is the amplitude scattering matrix (unitless).}$$

(exp(ikz) is incident plane wave)

For spheres $S_3 = S_4 = 0$ and $S_2(0°) = S_1(0°) = S(0°) = \frac{1}{2}\sum_n (2n+1)(a_n + b_n)$

Optical theorem: $\frac{4\pi}{k^2}\Re\{S(0°)\}$

Back to phase matrix: $\begin{pmatrix} I_{sca} \\ Q_{sca} \\ U_{sca} \\ V_{sca} \end{pmatrix} = \frac{C_{sca}}{4\pi R^2}\begin{pmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & -P_{34} & P_{44} \end{pmatrix}\begin{pmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{pmatrix}$

For spheres $P_{22} = P_{11}$ and $P_{44} = P_{33}$.

The off diagonal terms are usually small for Mie scattering, so polarization does not affect irradiance; need only P_{11} for I.

$$P_{11}(\Theta) = \frac{4\pi k^2}{C_{sca}}\frac{|S_1|^2 + |S_2|^2}{2}$$

Mie Scattering Amplitudes

$$S_1(\Theta) = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}\left[a_n\pi_n(\cos\Theta) + b_n\tau_n(\cos\Theta)\right]$$

$$S_2(\Theta) = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}\left[b_n\pi_n(\cos\Theta) + a_n\tau_n(\cos\Theta)\right]$$

The complex Mie coefficients a_n and b_n are obtained from matching the boundary conditions at the surface of the sphere. They are expressed in terms of spherical Bessel functions evaluated at x and mx.

The Mie angular functions:

$\pi_n(\cos\Theta) = \frac{1}{\sin\Theta}P_n^l(\cos\Theta)$

$\tau_n(\cos\Theta) = \frac{d}{d\Theta}P_n^l(\cos\Theta)$; P_n^l are associate Legendre functions.

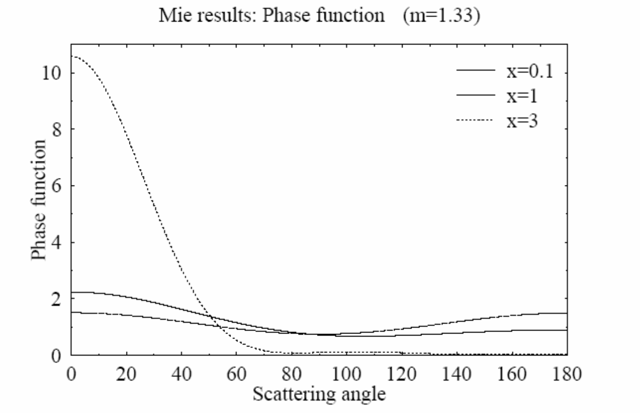
Mie Code Algorithm

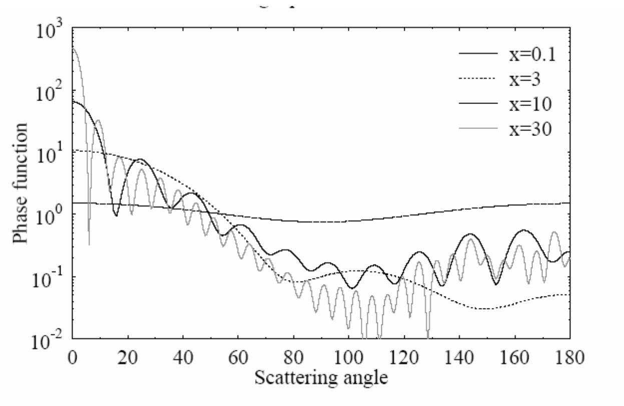
How a Mie code works:

1. Compute a_n and b_n for $n = 1 \ldots N$ from size parameter x and index of refraction m (uses recursion relations for the spherical Bessel functions). $N \approx x + 4x^{1/3} + 2$.
2. Compute Q_{ext}, Q_{sca}, and g from a_n and b_n.
3. Compute $S_1(\Theta)$ and $S_1(\Theta)$ at desired scattering angles from a_n and b_n and $\pi_n(\Theta)$ and $\tau_n(\Theta)$. Compute phase matrix elements P_{11}, P_{12}, P_{33}, P_{34} from S_1, S_2.
4. Integrate numerically over a size distribution $n(r)$ to get volume extinction β, single scattering albedo ϖ_0, and phase function $P(\Theta)$.

Mie Scattering Results: Phase Functions

Forward peak increases dramatically with x. ***For single particles*** - number of oscillations in $P(\Theta)$ increases with x.


Mie results: Phase function (m=1.33)



Extinction Paradox and Geometric Optics Limit

- Geometric optics limit: $x \rightarrow \infty$.
- As $x \rightarrow \infty$ extinction efficiency ~ 2 so extinction cross section is twice particle area.
  - One πr^2 from blockage by particle (scattered and absorbed), second πr^2 from diffraction (near-forward scattered at edge).
- Light diffracted by particle edge is scattered in small angles from incident direction.
- Solution to paradox: need to be in far field ($xr >> 1$) to see diffraction.

Diffraction peak in phase function (for geometric optics limit): the angular width of the forward scattering peak (defined at half the maximum, at $P(0°)$) occurs at $\Theta \approx 1/x$. The height at $P(0°) \propto x^2$.

A Mie Scattering Code and Lab

Download MieLab.zip from course web page

Files:

1. lab exercise file: mielab.pdf
2. <u>Fortran</u> source for Mie code: miegamma.f
3. <u>Fortran</u> source for index of refraction code: waterindex.f
4. <u>IDL</u> phase function plotting file: plotmie.pro
5. <u>IDL</u> procedure to read phase function file: readphase.pro
6. A Matlab mie code: matlab mie.pdf

You will need to know the gamma distribution:

$n(r) = \frac{Nb^{\alpha+1}}{\Gamma(\alpha+1)} r^{\alpha} \exp(-br)$ where $r_c = \alpha/b$ is the modal radius and α controls the width.

Moments: $\int_0^\infty r^k n(r) dr = Nb^{-k} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)}$