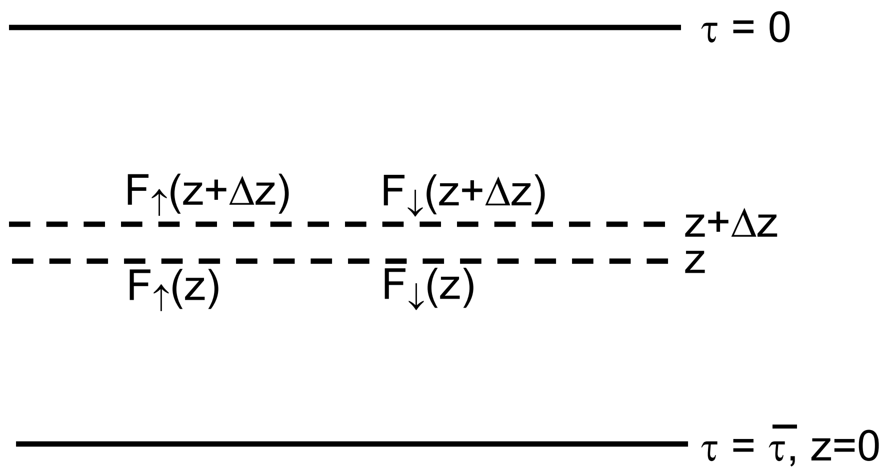
# Two-Stream Theory of Radiative Transfer

- radiation field consists of irradiances *F* in two and only two directions (streams), denoted as upward and downward.
- a photon directed downward can be scattered only downward or upward; similarly for a photon directed upward.




Downward ($F_{\downarrow}$) and upward ($F_{\uparrow}$) irradiances are different at z and $z+\Delta z$ because of absorption and scattering within $\Delta z$. The positive z-axis is downward.

# Two-Stream Theory of Radiative Transfer

- Conservation of upward radiant energy:

$$F_\uparrow(z+\Delta z) = F_\uparrow(z) - \beta_a \Delta z F_\uparrow(z) - \beta_s \Delta z p_{\uparrow\downarrow} F_\uparrow(z) + \beta_s \Delta z p_{\downarrow\uparrow} F_\downarrow(z+\Delta z)$$

- At top of layer the upward radiation is:
  - the incident radiation from below decreased by absorption and by scattering downward in $\Delta z$
  - *increased* because downward radiation incident at the top of the layer is scattered upward in $\Delta z$
- quantity $p_{\uparrow\downarrow}$ is the (conditional) probability that given that an upward photon is scattered, it is scattered in the downward direction, and similarly for $p_{\downarrow\uparrow}$.
- Divide both by $\Delta z$ and take the limit as $\Delta z \to 0$:

$$\frac{dF_\uparrow}{dz} = -\beta_a F_\uparrow - \beta_s p_{\uparrow\downarrow} F_\uparrow + \boxed{\beta_s p_{\downarrow\uparrow} F_\downarrow}$$

# Two-Stream Theory of Radiative Transfer

- Conservation of downward radiant energy:

$$F_\downarrow(z) = F_\downarrow(z+\Delta z) - \beta_a \cdot -\Delta z \cdot F_\downarrow(z+\Delta z) - \beta_s \cdot -\Delta z \cdot p_{\downarrow\uparrow} F_\downarrow(z+\Delta z) + \beta_s(-\Delta z) p_{\uparrow\downarrow} F_\uparrow(z)$$

  note sign change of attenuation of downward radiation in the direction of decreasing z

- At bottom of layer the downward radiation is:
  - the incident radiation at the top decreased by absorption and by scattering downward in $\Delta$z
  - *increased* because upward radiation incident at the bottom of the layer is scattered downward in $\Delta z$
- quantity $p_{\downarrow\uparrow}$ is the (conditional) probability that given that a down ward photon is scattered, it is scattered in the upward direction.

- Divide both by $\Delta$z and take the limit as $\Delta z \to 0$:

$$\frac{dF_\downarrow}{dz} = \beta_a F_\downarrow + \beta_s p_{\downarrow\uparrow} F_\downarrow - \beta_s p_{\uparrow\downarrow} F_\uparrow$$

note sign change of attenuation of downward radiation in the direction of decreasing z

# Two-Stream Theory of Radiative Transfer

- photons can be scattered only upward or downward:

$$p_{\downarrow\uparrow} + p_{\downarrow\downarrow} = p_{\uparrow\downarrow} + p_{\uparrow\uparrow} = 1$$

gain

$$\frac{dF_\uparrow}{dz} = -(\beta_a + \beta_s)F_\uparrow + \beta_s(p_{\uparrow\uparrow}F_\uparrow + p_{\downarrow\uparrow}F_\downarrow) \qquad \frac{dF_\downarrow}{dz} = (\beta_a + \beta_s)F_\downarrow - \beta_s(p_{\downarrow\downarrow}F_\downarrow + p_{\uparrow\downarrow}F_\uparrow)$$

loss

- assume that the ***medium*** is *isotropic*: $p_{\downarrow\uparrow} = p_{\uparrow\downarrow}$ and $p_{\downarrow\downarrow} = p_{\uparrow\uparrow}$
- Define the asymmetry parameter *g* as the mean cosine of the scattering angle:

$$\mathrm{g} = p_{\downarrow\downarrow}(1) + p_{\downarrow\uparrow}(\text{-}1) \text{ so } p_{\downarrow\uparrow} = p_{\uparrow\downarrow} = \frac{1-\mathrm{g}}{2} \text{ and } p_{\downarrow\downarrow} = p_{\uparrow\uparrow} = \frac{1+\mathrm{g}}{2}$$

- Finally optical depth:

$$\tau = -\int_0^z (\beta_a + \beta_s)dz = \tau_a + \tau_s$$

and single scattering albedo:

$$\varpi = \beta_s/(\beta_s + \beta_a)$$

# Two-Stream Theory of Radiative Transfer

$$\frac{dF_{\downarrow}}{d\tau} = -F_{\downarrow} + \varpi\left\{\frac{1+\mathrm{g}}{2}F_{\downarrow} + \frac{1-\mathrm{g}}{2}F_{\uparrow}\right\}$$

$$\frac{dF_{\uparrow}}{d\tau} = F_{\uparrow} - \varpi\left\{\frac{1+\mathrm{g}}{2}F_{\uparrow} + \frac{1-\mathrm{g}}{2}F_{\downarrow}\right\}$$

$$\frac{d}{d\tau}\left(F_{\downarrow} - F_{\uparrow}\right) = -(1-\varpi)\left(F_{\downarrow} + F_{\uparrow}\right)$$

$$\frac{d}{d\tau}\left(F_{\downarrow} + F_{\uparrow}\right) = -(1-\varpi g)\left(F_{\downarrow} - F_{\uparrow}\right)$$

# Conservative Scattering

no absorption: $\varpi = 1$

- assume that *g* is independent of $\tau$

$$F_{\downarrow} = B + C(1 - \tau^*); F_{\uparrow} = B - C(1 + \tau^*); \tau^* = (1 - g)\tau$$

- $\tau^*$ is called *scaled* optical thickness, meaning that it is scaled to isotropic scattering

Conservative Scattering: Equilibrium Solution *(infinitely thick medium)*.

Solution: C = 0 and $F_{\downarrow} = F_{\uparrow}$ = constant everywhere; same solution for finite layer above a perfectly reflecting mirror.

# Conservative Scattering

Conservative Scattering: Reflection and Transmission

- More realistic boundary conditions: $F_{\uparrow}(\tau = \bar{\tau}) = 0, F_{\downarrow}(\tau = 0) = F_o$

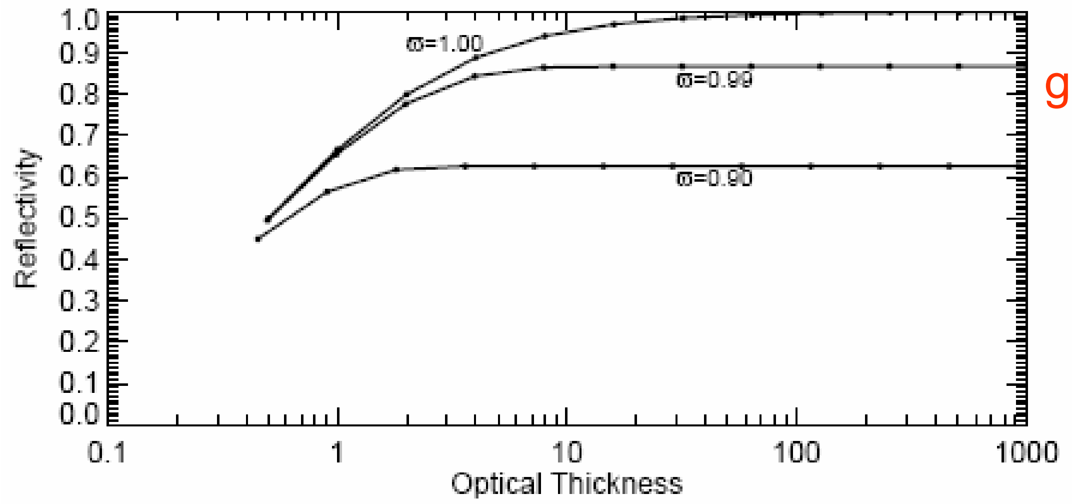
$$R = \frac{F_{\uparrow}(\tau = 0)}{F_o} = \frac{\bar{\tau}*/2}{1+\bar{\tau}*/2} = \frac{\bar{\tau}(1-g)/2}{1+\bar{\tau}(1-g)/2}$$

$$R = \frac{\bar{\tau}}{2/(1-g)+\tau} \approx \frac{\bar{\tau}}{13+\bar{\tau}}$$ for g = 0.85

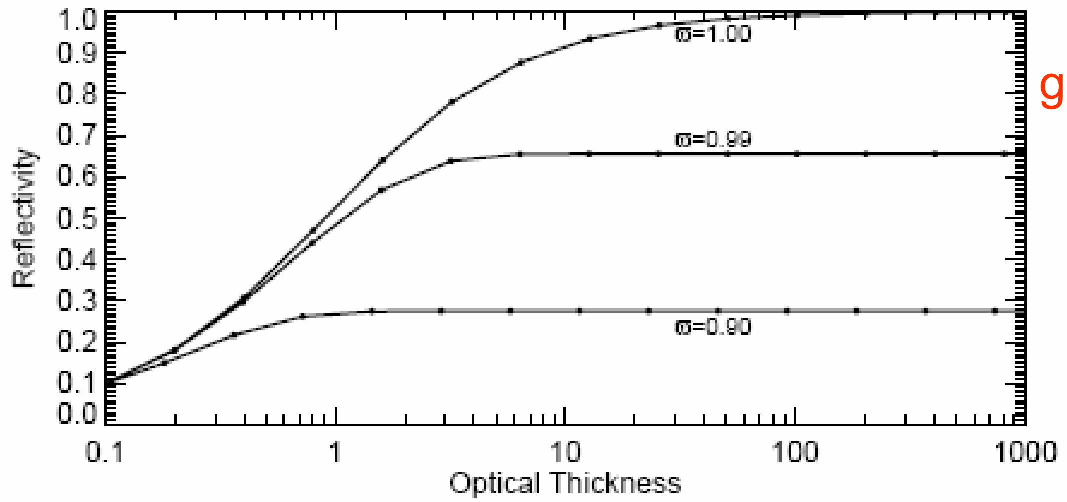
$$T = \frac{F_{\downarrow}(\bar{\tau})}{F_o} = \frac{1}{1+\bar{\tau}*/2} = \frac{1}{1+\bar{\tau}(1-g)/2} = \frac{1}{1+\bar{\tau}p_{\downarrow\uparrow}}$$

*attenuation of incident light is a consequence only of downward photons scattered upward; downward photons scattered downward continue to contribute tot he downward irradiance.*

# R approaches 1 for $\varpi = 0$



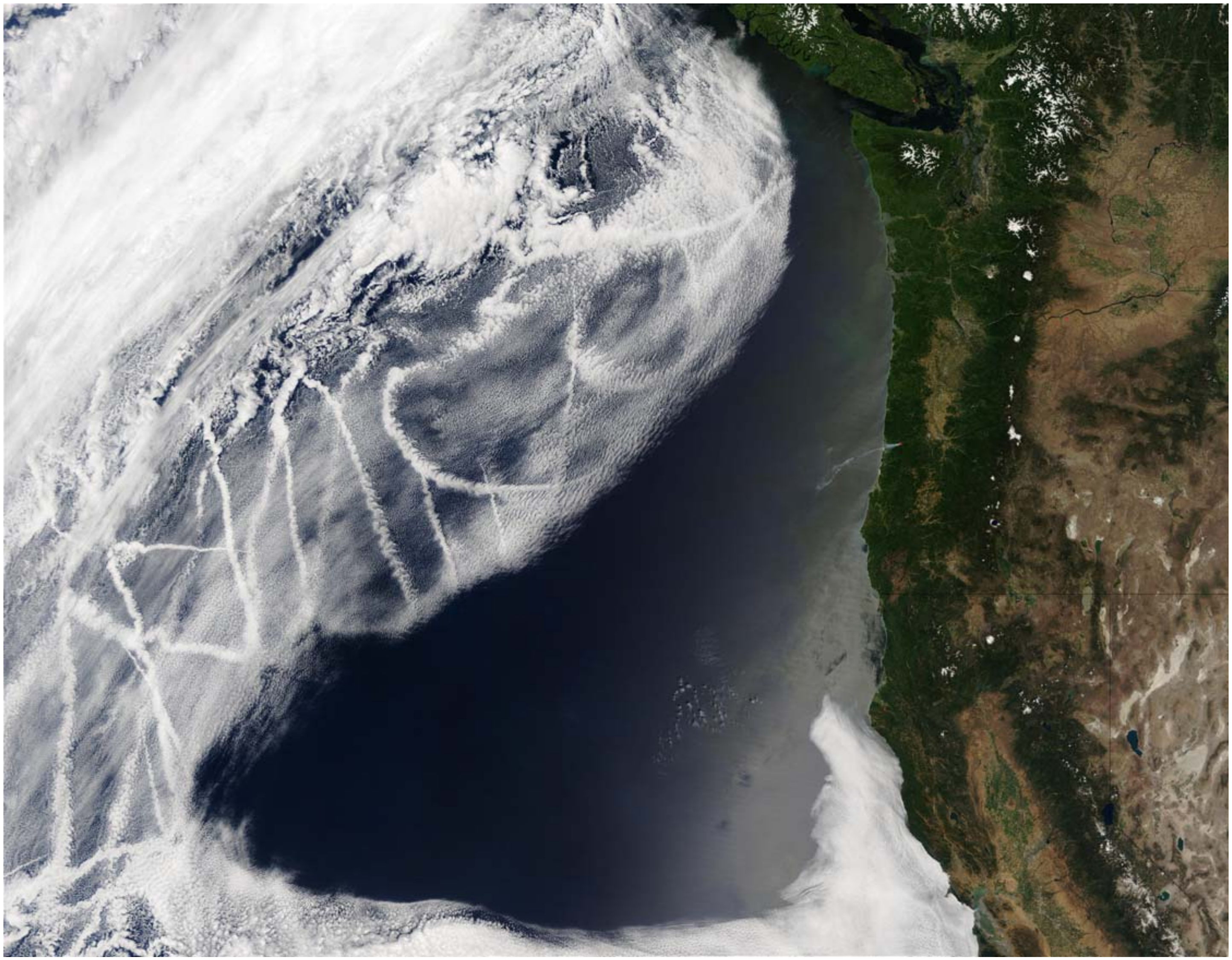

g = 0




g = 0.8

Aerosol Indirect Effect:

## Ship tracks off the NW US coast



# Problem

By how much does the scattering optical thickness of the cloud change at visible wavelengths if the effective radius decreases from 12 to 10 μm?

$$\tau = \int \beta dz = \beta \int dz = NC_{ext}\Delta z = Q_{ext}\pi r^2 N\Delta z \approx 2\pi r^2 N\Delta z$$

assuming scattering coefficient β independent of z and no absorption. We have also assumed that because *r/λ >> 1* extinction efficiency $Q_{ext} \approx 2$. $\Delta z$ is vertical depth of cloud, N is number density of droplets or raindrops. Use subscripts d and r to denote cloud droplets and raindrops, respectively:

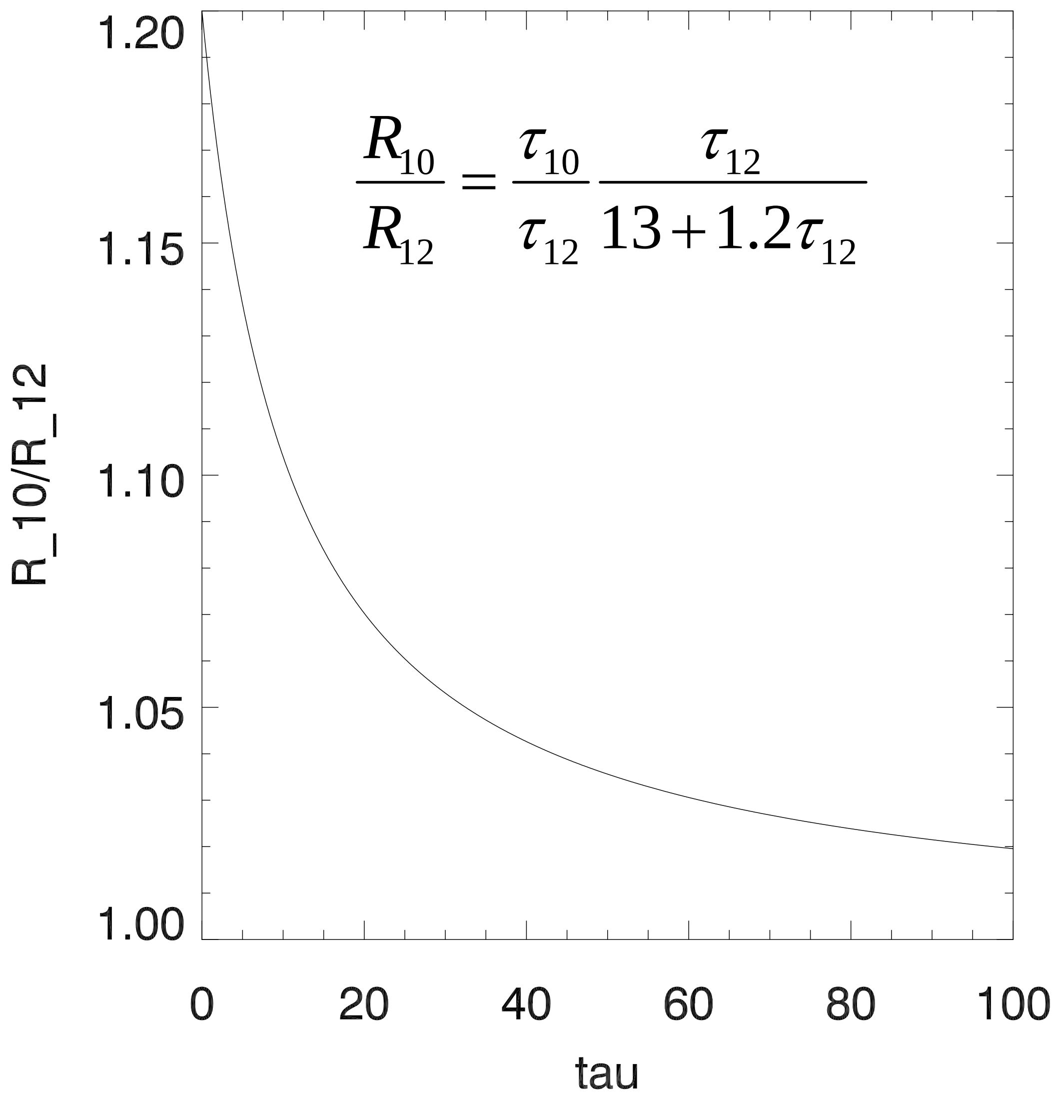
$$\frac{\tau_{10}}{\tau_{12}} = \frac{2\pi r_{10}^2 N_{10}\Delta z}{2\pi r_{12}^2 N_{12}\Delta z} = \frac{r_{10}^2 N_{10}}{r_{12}^2 N_{12}}$$

Since we age keeping total condensed fixed water the liquid water (mass water/volume air) content for both droplets and raindrops is the same:

$$LWC_{10} = \frac{4}{3}\pi r_{10}^3 \rho N_{10} = LWC_{12} = \frac{4}{3}\pi r_{12}^3 \rho N_{12}$$

So: $$\frac{N_{10}}{N_{12}} = \frac{r_{12}^3}{r_{10}^3} \text{ and } \frac{\tau_{10}}{\tau_{12}} = \frac{r_{12}}{r_{10}} = 1.2$$

By how much does will the reflectance of the cloud change at visible wavelengths if the effective radius decreases from 12 to 10 μm?




$$\frac{R_{10}}{R_{12}} = \frac{\tau_{10}}{\tau_{12}} \frac{\tau_{12}}{13 + 1.2\tau_{12}}$$

# The aerosol "indirect effect"

Cloud optical thickness: $\tau = 2\pi r^2 N \Delta z$

Cloud liquid water content: $LWC = \frac{4}{3}\pi r^3 \rho N = \frac{2}{3}\rho r \tau / \Delta z$

Sensitivity of $\tau$ to N, fixed W: $\frac{d\tau}{\tau} = \frac{dN}{3N}$

Two stream approximation: $R = \frac{\tau}{(8 \text{ to } 13)^* + \tau}$ * for g=0.75 to 0.85

Sensitivity of R to N, fixed W: $\frac{dR}{dN} = \frac{dR}{d\tau}\frac{d\tau}{dN} = \frac{R(1-R)}{3N}$

# Cloud "Susceptibility"

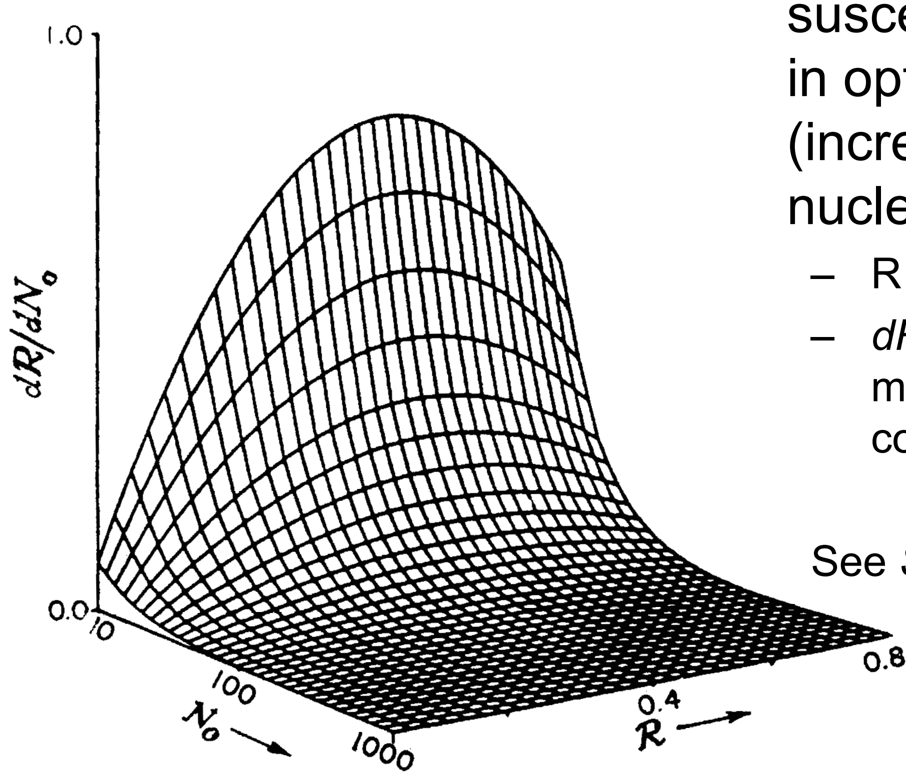



**Figure 6.20** The susceptibility parameter $d\mathcal{R}/dN_o$ for different conditions.

*dR/dN* is a measure of the susceptibility of clouds to changes in optical depth due to pollution (increases in cloud condensation nuclei)

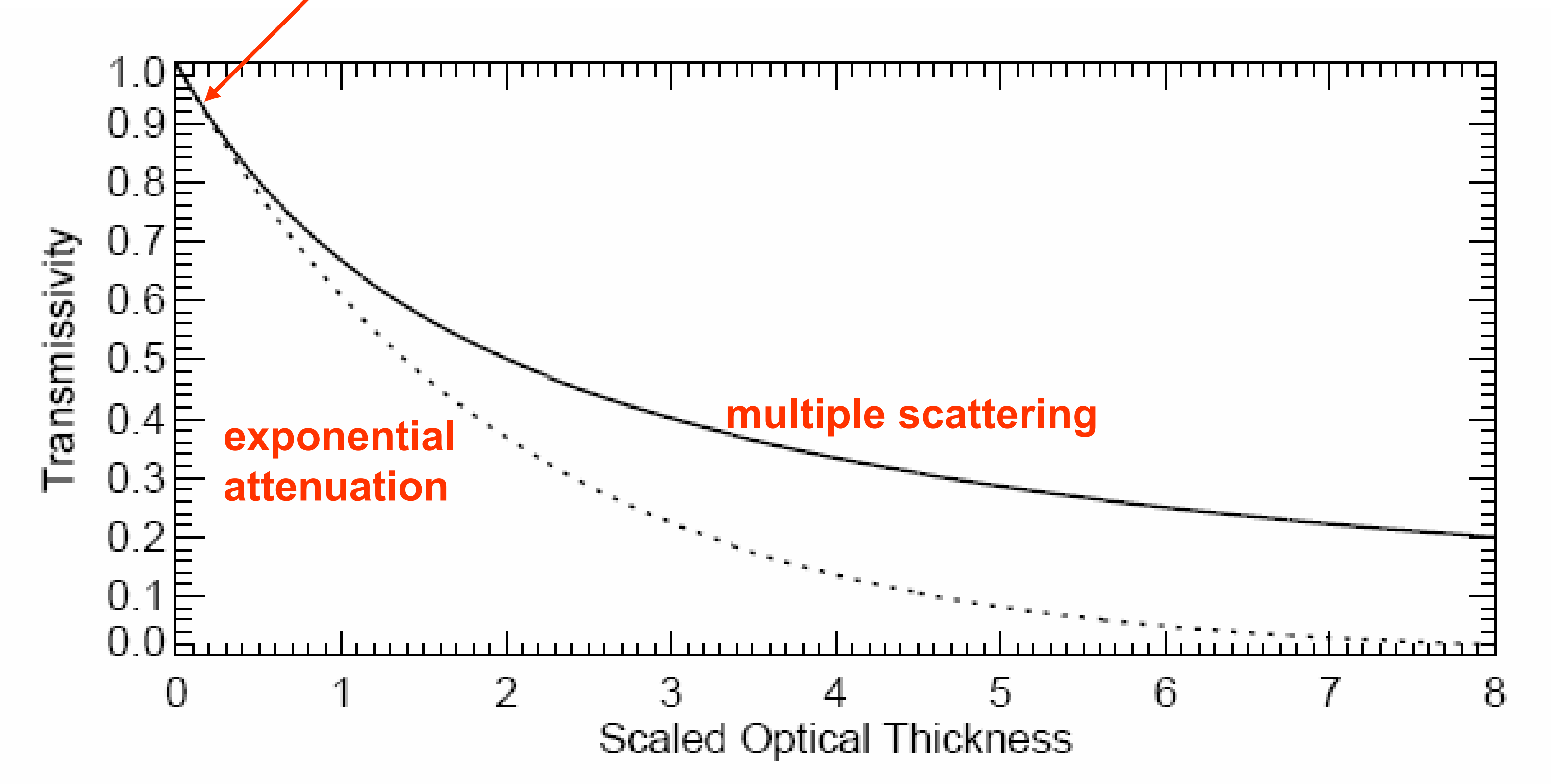
- R = 0.5: most susceptible
- *dR/dN* is inversely proportional to N: most susceptible under cleanest conditions

See *Stephens*, pp. 312-313, 219-221

# Conservative Scattering

Transmissivity $T = \dfrac{F_{\downarrow}(\bar{\tau})}{F_o} = \dfrac{1}{1+\bar{\tau}^*/2}$

$$\bar{\tau}^* << 1:\ \mathrm{T} \approx 1 - \bar{\tau}^*/2 \approx \exp\left(-\bar{\tau}^*/2\right)$$



# Diffuse Radiation

- Consider layer with irradiance $F_0$ incident at top.
- The irradiance of the light that has not been scattered (superscript "u" is for un-scattered) is an exponential function of optical depth into the medium:

$$F_{\downarrow}^{u} = F_0 \exp(-\tau)$$

- diffuse downward irradiance ($D_{\downarrow}$) is light that has been scattered:

$$D_{\downarrow} = F_{\downarrow} - F_{\downarrow}^{u}$$

- Since all upward irradiance is scattered irradiance, $D_{\uparrow} = D_{\uparrow}$

# Diffuse Radiation

- Recall: $F_{\downarrow} = B + C(1-\tau^*); F_{\uparrow} = B - C(1+\tau^*); \tau^* = (1-g)\tau$
- At bottom of layer ($\tau = \bar{\tau}$) $D_{\uparrow} = 0$

$$\frac{D_{\downarrow}}{F_0} = \frac{1 + \left(\bar{\tau}^* - \tau^*\right)/2}{1 + \bar{\tau}^*/2} - \exp(-\tau)$$

$$\frac{D_{\uparrow}}{F_0} = \frac{\left(\bar{\tau}^* - \tau^*\right)/2}{1 + \bar{\tau}^*/2}$$

Assume optically thick medium: $\bar{\tau}(1-g)/2 >> 1$

For Exponential can be ignored for $\tau$ > *2/(1-g)*

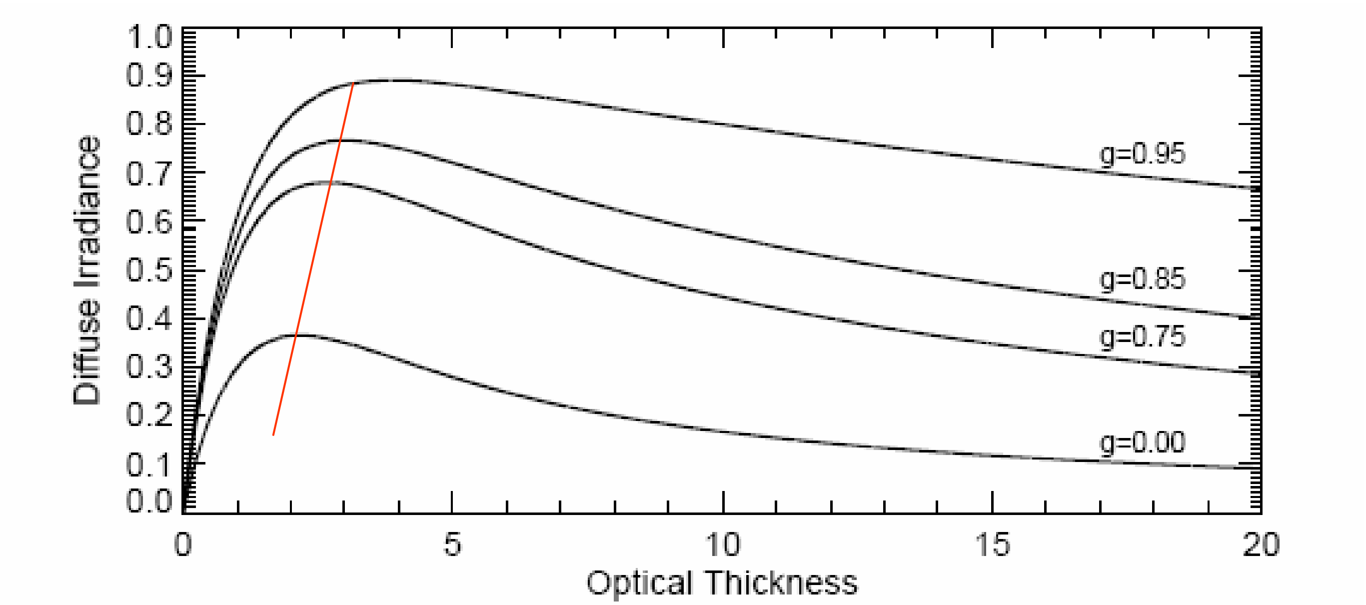
$D_{\downarrow}$ = $D_{\uparrow}$ when $\left(\bar{\tau}^* - \tau^*\right)/2 > 1$ or $\tau^* < \bar{\tau}^* - 2$

If we are further than *2/(1-g)* away from cloud boundaries the radiation field is isotropic. (since $\tau p_{\downarrow\uparrow} = \tau p_{\uparrow\downarrow} = \tau(1-g)/2 = 1$ it is virtually certain that a photon has been turned around*)*

# Diffuse Radiation

Diffuse irradiance beneath a cloud: $\tau = \overline{\tau}$

$$\frac{D_{\downarrow}}{F_0} = \frac{1}{1+\overline{\tau}^{*}/2} - \exp(-\overline{\tau}) \qquad \overline{\tau}_{\max} = (1/g)\ln[2/(1-g)]$$



# Problem

Estimate the optical thickness of a cloud such that its transmitted (diffuse) radiance is equal to the attenuated (by the cloud) radiance of direct sunlight. This is an estimate of how optically thick a cloud must be such that the disc of the sun cannot be seen through it.

HINT: Assume that the cloud is sufficiently thick that the downward radiance emerging from the cloud is isotropic.

# Diffuse Radiation

Diffuse irradiance beneath a cloud:

$$D_{\downarrow} = F_0\left[\frac{1}{1+\bar{\tau}^* / 2} - \exp(-\bar{\tau})\right] = L_0\Omega\left[\frac{1}{1+\bar{\tau}^* / 2} - \exp(-\bar{\tau})\right]$$

Where $L_0$ is solar radiance and $\Omega$ is solid angle of sun.

Assuming isotropic scattering, diffuse radiance, $L_d$ is:

$$L_d = \frac{D_{\downarrow}}{\pi} = \frac{L_0\Omega}{\pi}\left[\frac{1}{1+\bar{\tau}^* / 2} - \exp(-\bar{\tau})\right] = L_0 \exp(-\bar{\tau})$$

Where the right hand side of the equation sets the criterions when the solar disk can no longer be distinguished. Rearranging (recognizing that $\pi/\Omega >> 1$):

$$\frac{2}{2+\bar{\tau}(1-g)} \approx 5\times 10^4 \exp(-\bar{\tau})$$

Solve iteratively, taking g = 0.85 for clouds: $10 < \bar{\tau} < 11$

# Multiple Scattering in an Absorbing Medium

$$\frac{d}{d\tau}\left(F_{\downarrow} - F_{\uparrow}\right) = -(1-\varpi)\left(F_{\downarrow} + F_{\uparrow}\right) \qquad \frac{d}{d\tau}\left(F_{\downarrow} + F_{\uparrow}\right) = -(1-\varpi g)\left(F_{\downarrow} - F_{\uparrow}\right)$$

$$\frac{d^2}{d\tau}\left(F_{\downarrow} + F_{\uparrow}\right) = -(1-\varpi g)\frac{d}{d\tau}\left(F_{\downarrow} - F_{\uparrow}\right) = (1-\varpi g)(1-\varpi)\left(F_{\downarrow} + F_{\uparrow}\right)$$

$$\frac{d^2}{d\tau}\left(F_{\downarrow} - F_{\uparrow}\right) = -(1-\varpi)\frac{d}{d\tau}\left(F_{\downarrow} + F_{\uparrow}\right) = (1-\varpi)(1-\varpi g)\left(F_{\downarrow} - F_{\uparrow}\right)$$

$$\frac{d^2 F}{d\tau} = K^2 F \text{ where } K = \sqrt{(1-\varpi g)(1-\varpi)}$$

solution of the form: $A\exp(\pm K\tau)$

*F* is difference or sum of upward and downward irradiance.

For infinite layer, $F_0$ incident from above:

$$F_{\downarrow} = F_0 \exp\left(-K\tau\right); F_{\uparrow} = F_0 R_{\infty} \exp\left(-K\right)$$

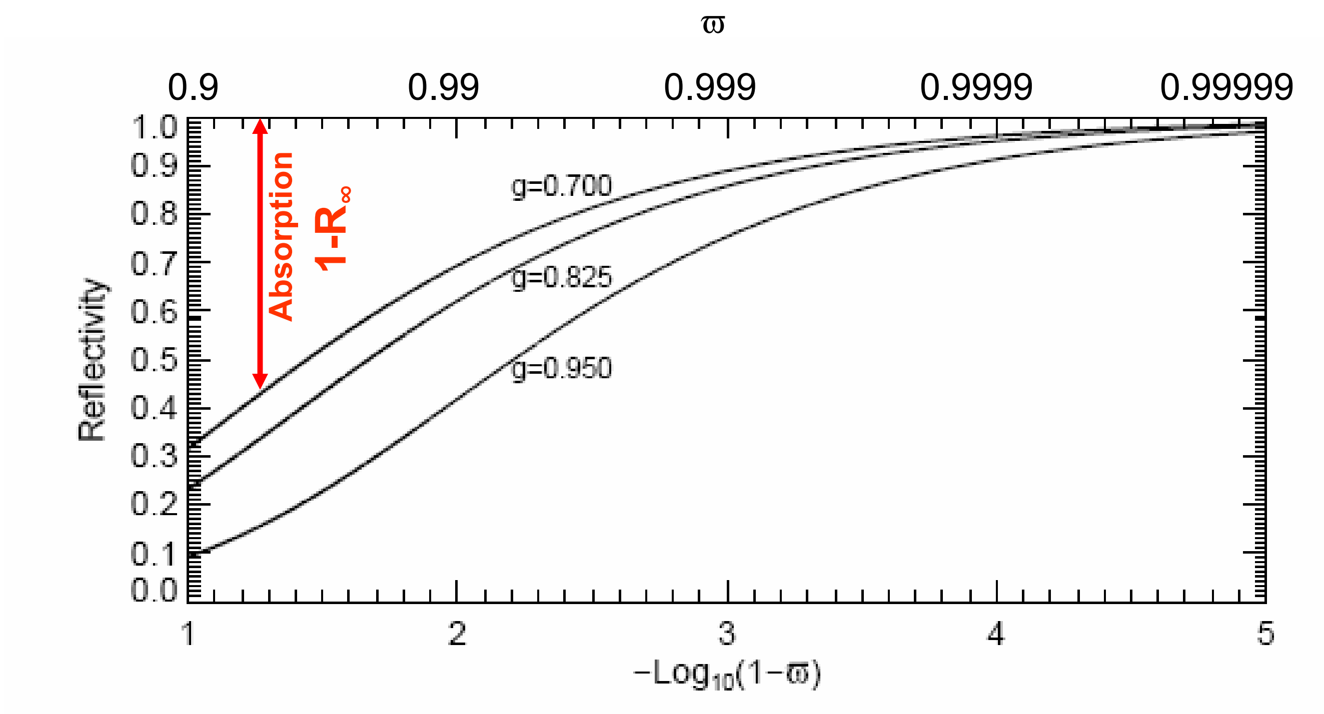
$$R_{\infty} = \frac{\sqrt{1-\varpi g} - \sqrt{1-\varpi}}{\sqrt{1-\varpi g} + \sqrt{1-\varpi}}$$

# Reduction in reflectance by multiple scattering

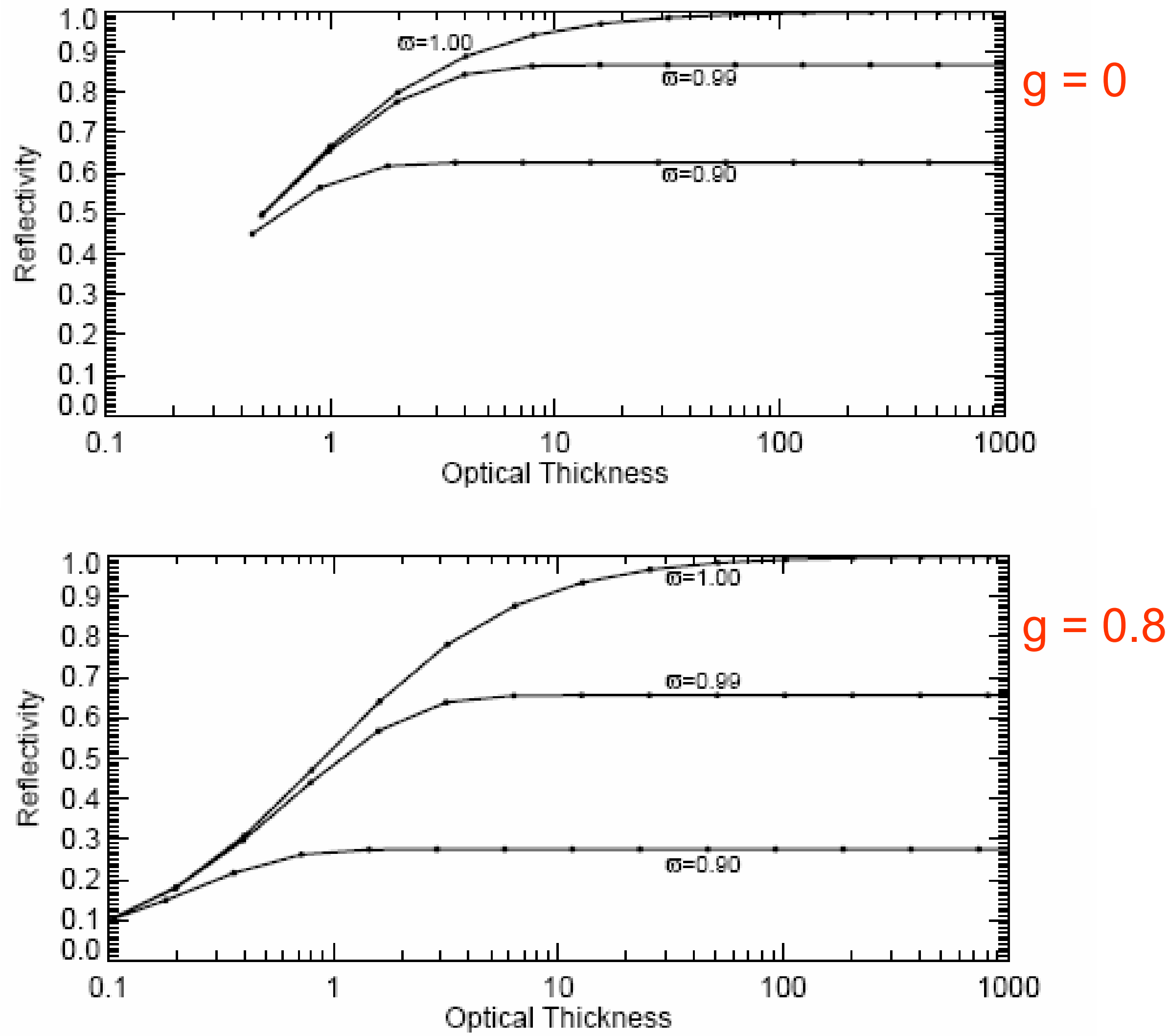
For weak absorption: $R_\infty \approx 1 - 2\sqrt{\dfrac{\beta_a}{\beta_s(1-g)}} \approx 1 - 2\sqrt{\dfrac{1-\varpi}{1-g}}$

Recall (in ch. 3) that for weak absorption (κh <<1) $C_{abs}$ ≈ volume*κ and (ch. 3) for particles much larger than wavelength of light $C_{sca}$ ≈ 2*area.

So $$1-\varpi = \frac{\beta_a}{\beta_a+\beta_s} \approx \beta_a / \beta_s = C_{abs} / C_{sca} \propto r\beta_a \text{ so } 1-R_\infty \propto \sqrt{r\beta_a}$$

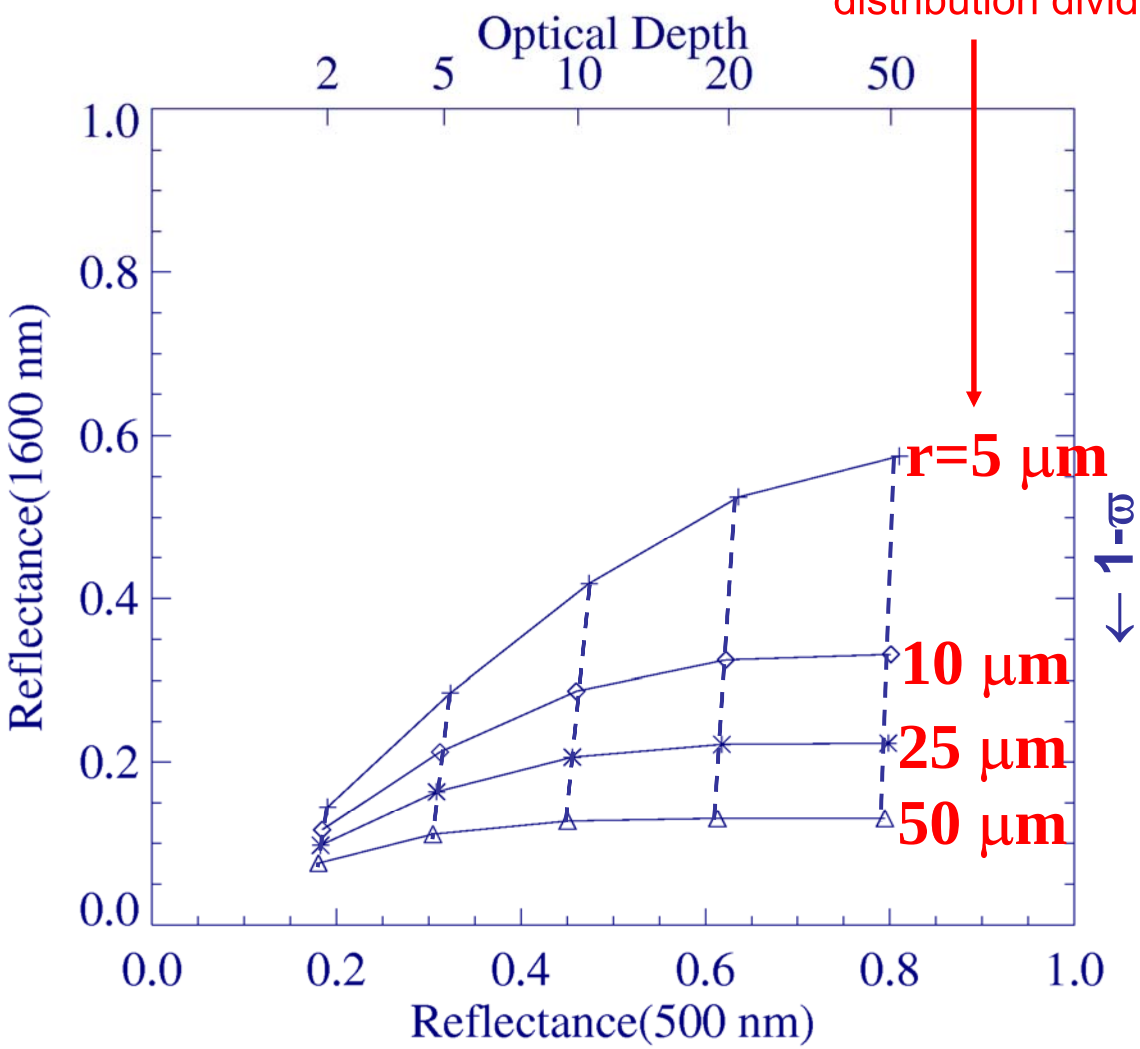


# R approaches 1 for $\varpi = 0$



# Cloud Reflectance

These are effective radii: third moment of size distribution dividend by second moment




This is how cloud droplet size is derived from measured cloud reflectance. In the visible, where water is virtually non absorbing, cloud reflectance is a function of optical depth. In the near-infrared, where liquid water and ice are weakly absorbing, cloud reflectance depends most strongly on $\varpi$, which we have seen can be directly related to droplet size.

# Reduction in reflectance by multiple scattering

Reflectance measured at 70 Kft from thick cirrus layer