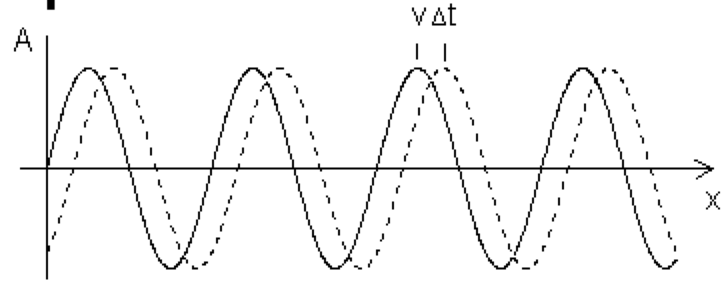
Phase speed and refractive index



- $n = c/v$, where v ($= \omega/k$) is phase speed; can $n < 1$?
- Does this violate special relativity?
  - No restrictions to speed of a plane wave. Signals cannot propagate faster than c.
- Plane harmonic wave cannot transmit a signal – requires modulating waves.
- Phase speed is neither the speed of a thing nor speed of a signal

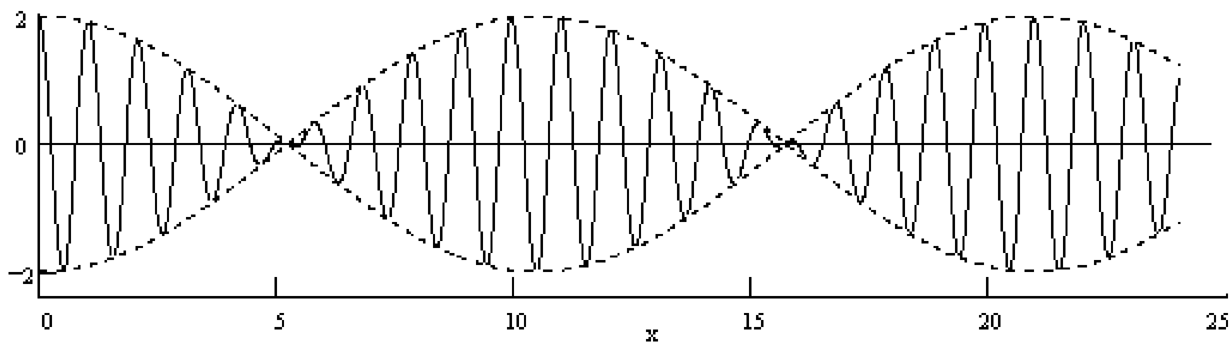
Group Velocity

- In general, there is no upper limit on the possible phase velocity of a wave.
- How about *group velocity*?
- consider two superimposed sine waves with equal amplitude a but with slightly different frequencies and wavenumbers (say $\omega \pm \Delta\omega$ and $k \pm \Delta k$):

$$\psi = \cos[(k - \Delta t)x - (\omega - \Delta\omega)t] + \cos[(k + \Delta t)x - (\omega + \Delta\omega)t] = 2\cos(kx - \omega t)\cos(\Delta kx - \Delta\omega t)$$

$$A(x,t) = \cos[(k-\Delta t)x - (\omega - \Delta\omega)t] + \cos[(k+\Delta t)x - (\omega + \Delta\omega)t] = 2\cos(kx - \omega t)\cos(\Delta kx - \Delta\omega t)$$

- This is a simple sinusoidal wave with angular velocity ω, wave number k, and modulated amplitude 2cos(Δkx - $\Delta\omega$t). In other words, the amplitude of the wave is itself a wave, and the phase velocity of this modulation wave is v = $\Delta\omega/\Delta$k.



- speed of modulating envelope is $\Delta\omega/\Delta k$. This is the phase velocity of the amplitude wave, but since each amplitude wave contains a group of internal waves, this speed is usually called the ***group velocity.***

- Physical waves of a given type in a given medium generally exhibit a characteristic group velocity as well as a characteristic phase velocity.
- Define group velocity:

$$\frac{1}{v_g} = \lim_{\Delta\omega \to 0} \frac{\Delta k}{\Delta \omega} = \frac{dk}{d\omega}$$

- Recall: $v = c/n = \omega/k$, $\omega = kc/n$: $v_g = \frac{1}{dk/d\omega} = \frac{c}{n + \omega(dn/d\omega)}$

- refractive index is a function of the frequency in dispersive media
- group velocity corresponds to the actual signal velocity only under conditions of normal dispersion, or, more generally, under conditions when the group velocity is less than the phase velocity
- in a regime of anomalous dispersion, which means the refractive index *decreases* with increasing wave number, the preceding formula shows that what we called the group velocity exceeds what we called the phase velocity. In such circumstances the group velocity no longer represents the speed at which information or energy propagates.

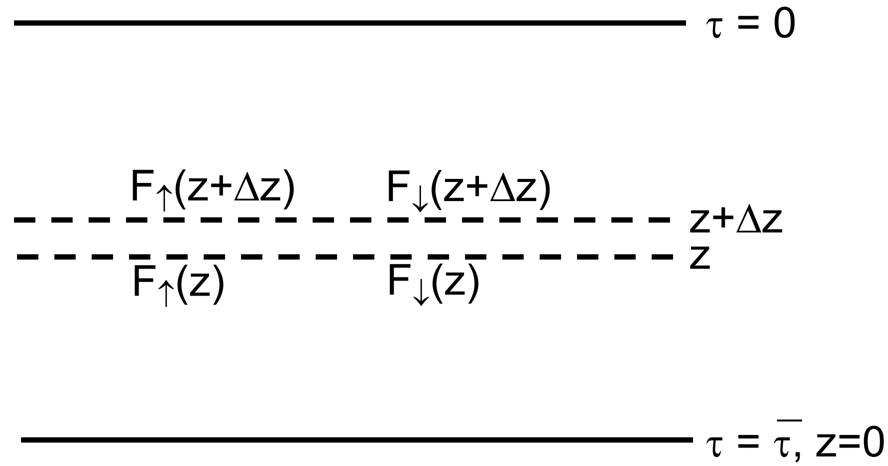
$$v_g = \frac{c}{n + \omega(dn/d\omega)} = \frac{v}{1 + (\omega/n)(dn/d\omega)}$$

- *normal dispersion* is when $dn/d\omega > 0$ (and $v_g < v$)
- *anomalous dispersion* when $dn/d\omega < 0$ (and $v_g > v$)
- *Suppose that* $n + \omega\frac{dn}{d\omega} < 0$ *and* $\left|n + \omega\frac{dn}{d\omega}\right| < 1$

  group velocity is negative and its magnitude is greater than *c*: light breaks the speed limit while traveling the wrong way!
- the signal speed is the speed with which signals are propagated
  - Suppose that a source in a dispersive medium is turned on
  - After time t a detector a distance d from the source registers a signal
  - d/t cannot be greater than c.

Two-Stream Theory of Radiative Transfer

- radiation field consists of irradiances *F* in two and only two directions (streams), denoted as upward and downward.
- a photon directed downward can be scattered only downward or upward; similarly for a photon directed upward.



Downward ($F_\downarrow$) and upward ($F_\uparrow$) irradiances are different at z and $z+\Delta z$ because of absorption and scattering within Δz. The positive z-axis is downward.

Two-Stream Theory of Radiative Transfer

- Conservation of upward radiant energy:

$$F_{\uparrow}(z+\Delta z) = F_{\uparrow}(z) - \beta_a \Delta z F_{\uparrow}(z) - \beta_s \Delta z p_{\uparrow\downarrow} F_{\uparrow}(z) + \beta_s \Delta z p_{\downarrow\uparrow} F_{\downarrow}(z+\Delta z)$$

- At top of layer the upward radiation is:
  - the incident radiation from below decreased by absorption and by scattering upward in Δz
  - *increased* because downward radiation incident at the top of the layer is scattered upward in Δz
- quantity $p_{\uparrow\downarrow}$ is the (conditional) probability that given that an upward photon is scattered, it is scattered in the downward direction, and similarly for $p_{\downarrow\uparrow}$.
- Divide both by Δz and take the limit as $\Delta z \rightarrow 0$:

$$\frac{dF_{\uparrow}}{dz} = -\beta_a F_{\uparrow} - \beta_s p_{\uparrow\downarrow} F_{\uparrow} + \boxed{\beta_s p_{\downarrow\uparrow} F_{\downarrow}}$$

Two-Stream Theory of Radiative Transfer

- Conservation of downward radiant energy:

  $$F_{\downarrow}(z) = F_{\downarrow}(z+\Delta z) - \beta_a \cdot -\Delta z \cdot F_{\downarrow}(z+\Delta z) - \beta_s \cdot -\Delta z \cdot p_{\downarrow\uparrow} F_{\downarrow}(z+\Delta z) + \beta_s \cdot -\Delta z \cdot p_{\uparrow\downarrow} F_{\uparrow}(z)$$

  note sign change of attenuation of downward radiation in the direction of decreasing z
- At bottom of layer the downward radiation is:
  - the incident radiation at the top of the layer is decreased by absorption and by scattering downward in Δz
  - *increased* because upward radiation incident at the bottom of the layer is scattered downward in Δz
- quantity $p_{\downarrow\uparrow}$ is the (conditional) probability that given that a downward photon is scattered, it is scattered in the upward direction.

- Divide both by Δz and take the limit as $\Delta z \rightarrow 0$:

$$\frac{dF_{\downarrow}}{dz} = \beta_a F_{\downarrow} + \beta_s p_{\downarrow\uparrow} F_{\downarrow} - \beta_s p_{\uparrow\downarrow} F_{\uparrow}$$

note sign change of attenuation of downward radiation in the direction of decreasing z

Two-Stream Theory of Radiative Transfer

- photons can be scattered only upward or downward:

$$p_{\downarrow\uparrow} + p_{\downarrow\downarrow} = p_{\uparrow\downarrow} + p_{\uparrow\uparrow} = 1$$

gain

$$\frac{dF_\uparrow}{dz} = -(\beta_a + \beta_s)F_\uparrow + \beta_s(p_{\uparrow\uparrow}F_\uparrow + p_{\downarrow\uparrow}F_\downarrow) \qquad \frac{dF_\downarrow}{dz} = (\beta_a + \beta_s)F_\downarrow - \beta_s(p_{\downarrow\downarrow}F_\downarrow + p_{\uparrow\downarrow}F_\uparrow)$$

loss

- assume that the ***medium*** is *isotropic*: $p_{\downarrow\uparrow} = p_{\uparrow\downarrow}$ and $p_{\downarrow\downarrow} = p_{\uparrow\uparrow}$
- Define the asymmetry parameter *g* as the mean cosine of the scattering angle:

$$\mathrm{g} = p_{\downarrow\downarrow}(1) + p_{\downarrow\uparrow}(\text{-}1) \text{ so } p_{\downarrow\uparrow} = p_{\uparrow\downarrow} = \frac{1-\mathrm{g}}{2} \text{ and } p_{\downarrow\downarrow} = p_{\uparrow\uparrow} = \frac{1+\mathrm{g}}{2}$$

- Finally optical depth:

$$\tau = -\int_0^z (\beta_a + \beta_s)dz = \tau_a + \tau_s$$

and single scattering albedo:

$$\varpi = \beta_s / (\beta_s + \beta_a)$$

Two-Stream Theory of Radiative Transfer

$$\frac{dF_{\downarrow}}{d\tau} = -F_{\downarrow} + \varpi\left\{\frac{1+\mathrm{g}}{2}F_{\downarrow} + \frac{1-\mathrm{g}}{2}F_{\uparrow}\right\}$$

$$\frac{dF_{\uparrow}}{d\tau} = F_{\uparrow} - \varpi\left\{\frac{1+\mathrm{g}}{2}F_{\uparrow} + \frac{1-\mathrm{g}}{2}F_{\downarrow}\right\}$$

$$\frac{d}{d\tau}\left(F_{\downarrow} - F_{\uparrow}\right) = -(1-\varpi)\left(F_{\downarrow} + F_{\uparrow}\right)$$

$$\frac{d}{d\tau}\left(F_{\downarrow} + F_{\uparrow}\right) = -(1-\varpi g)\left(F_{\downarrow} - F_{\uparrow}\right)$$