Expansion of Phase Function in Legendre Series

Real phase functions are usually expressed as infinite series of orthogonal functions, usually Legendre polynomials (P_l): $p(\cos\Theta) = \sum_{l=0}^{\infty} \varpi_l P_l(\cos\Theta)$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = (3x^2 - 1)/2$
$P_3(x) = (5x^3 - 3x)/2$
.
.
.

Legendre polynomials are *orthogonal*:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0, n \neq m \\ \dfrac{2}{2n+1}, \text{n = m} \end{cases}$$

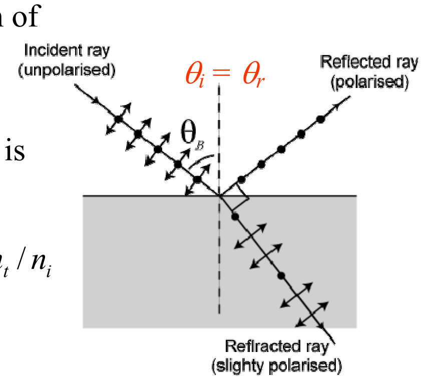
Recall: $g = \langle \cos\Theta \rangle = \dfrac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} p(\cos\Theta) \cos\Theta d(\cos\Theta) d\varphi$

Show that $g = \dfrac{\varpi_1}{3}$

$$g = \frac{1}{2}\int_{-1}^{1} p(x) x dx = \frac{1}{2}\int_{-1}^{1} P_1 \left\{ \sum_{l=0}^{\infty} \varpi_l P_l(x) \right\} dx = \frac{1}{3}\varpi_1$$

Polarization upon reflection

- Electric dipoles in the media oscillate in the direction of the polarization of the transmitted light.
- Dipoles do not radiate energy in the direction in which they oscillate.
- If the direction of the transmitted light is perpendicular to the direction of specular reflectance the dipoles will not create any reflected light.
- This occurs at $\theta_r + \theta_t = \pi/2 = \theta_i + \theta_t$ or $\theta_t = \pi/2 - \theta_i$
- Snel's law: $\sin\theta_i = \dfrac{n_t}{n_i} \sin\theta_t = m \sin\theta_t = m\cos\theta_i$ if the above condition is met.
- 100% polarization ($R_{\|} = 0$) occurs at the Brewster angle: $\tan\theta_i = m = n_t / n_i$
- For water in air, $m = 1.33$, $\theta_B = 53°$



Simple theory: rainbows and halos

- Geometric optics (x >>1): Angular distribution of scattered light can be derived from ray tracing
  - Snel's law of refraction
  - Fresnel formulas gives polarized reflection *amplitude* coefficients
- Geometric optics predicts singularities, called ***caustics***: points at which differential scattering cross-section for set of rays is infinite
- For both the rainbow and halo, caustics are associated with angles of minimum deviation of the scattering angle (with respect to the direction of incident beam):

$$\frac{d\Theta}{d\theta_i} = 0.$$

Θ is scattering angle and θ_i is angle of incidence.

Supernumerary Bows

- Cannot be described without invoking interference
- Except at the rainbow angle, a horizontal line intersects the curve of scattering angle versus angle of incidence in two points
- Two rays may have the same direction but follow different trajectories in a drop.
- The two waves corresponding to the two rays in the same direction are different in phase and hence interfere.
- Interference depends on drop size (unlike the positions of rainbow)

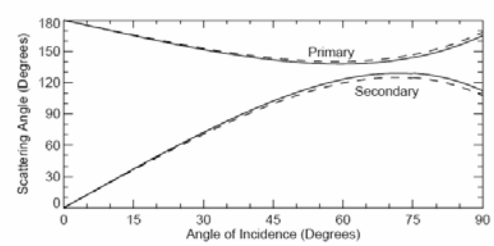
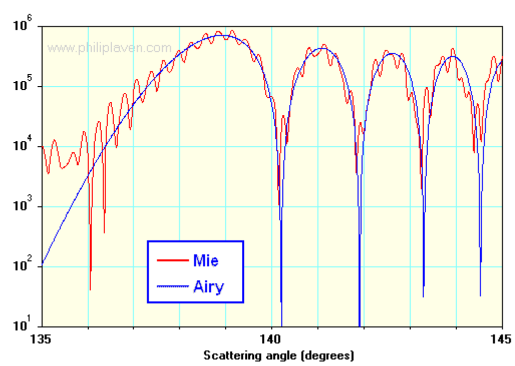


Figure 8.24: The scattering angle for incident rays undergoing one internal reflection by a large, transparent sphere is a minimum for a primary rainbow, whereas that for incident rays undergoing two internal reflections is a maximum for a secondary rainbow. The solid curve is for a wavelength of 650 nm, the dashed curve for 425 nm.



- George Biddell Airy's paper "On the intensity of light in the neighborhood of a caustic" was published in 1838.
- Showed that the intensity of light in a rainbow could be modeled using a cubic wave-front.

Supernumerary Bows

Equation of Transfer with Sources

Increase (due to emission and scattering) in radiance L_λ along path ds:

$$dL_\lambda = J_\lambda \beta ds \text{, where } J \text{ is the source function.}$$

Including extinction with source term:

$$dL_\lambda(s, \Omega) = \beta\,[-L_\lambda(s, \Omega) + J_\lambda(s, \Omega)]ds$$

$I\,(s, \Omega)$ is the radiance at point s along path in direction Ω, and β is the volume extinction coefficient.

Using $dz = \mu ds$ and $d\tau = -\beta dz$; change of sign because z increases upward and τ inc

$$dL_\lambda(z, \mu,\varphi) = [-L_\lambda(z,\mu,\varphi) + J_\lambda(z, \mu,\varphi)]\beta dz/\mu$$
$$dL_\lambda(\tau,\mu,\varphi) = [L_\lambda(\tau,\mu,\varphi) - J_\lambda(\tau,\mu,\varphi)]d\tau/\mu \text{ (note change in sign as above)}$$

Radiative Transfer Equation for Thermal Emission

- If there is no scattering then the absorptivity along path ds is $a_\lambda = \beta_\lambda ds$.
- In LTE Kirchhoff's Law says emissivity equals absorptivity: $\epsilon_\lambda = \beta_\lambda ds$.
- In LTE, thermal emission from path ds is $dL_\lambda = \epsilon_\lambda B_\lambda(T) = B_\lambda(T)\beta_\lambda\, ds$. Therefore, the source function equals the Planck function, $J = B_\lambda(T)$. (note: we will use B for the radiance form of the Planck function.)
- No azimuthal dependence because of isotropy of thermal emission.

$$\mu \frac{dL(\tau,\mu)}{d\tau} = L(\tau,\mu,) - B[T(\tau)]$$ (monochromatic but wavelength subscripts are omitted).

Multiply by $\frac{1}{\mu} e^{-\tau/\mu}$: $e^{-\tau/\mu} \frac{dL(\tau,\mu)}{d\tau} - \frac{1}{\mu} e^{-\tau/\mu} L(\tau,\mu) = -\frac{1}{\mu} e^{-\tau/\mu} B(T)$

$$\frac{d}{d\tau}\left(e^{-\tau/\mu} L(\tau,\mu) \right) = -\frac{1}{\mu} e^{-\tau/\mu} B(T)$$

Boundary conditions: Surface emits as blackbody ($T = T_0$ at $\tau = \tau_0$) and there is no downward irradiance at the top of atmosphere:

$$L(\tau_0, \mu > 0) = B(T_0) \text{ and } L(0, \mu < 0) = 0$$

$$\left[e^{-\tau/\mu} L(\tau, \mu > 0) \right]_\tau^{\tau_o} = -\int_\tau^{\tau_0} e^{-\tau'/\mu} B(T) \frac{d\tau'}{\mu}$$

$L(\tau, \mu > 0) = e^{-(\tau_0 - \tau)/\mu} B(T_0) + \int_\tau^{\tau_0} e^{-(\tau' - \tau)/\mu} B(T) \frac{d\tau'}{\mu}$; τ' - τ is optical depth from observer to integrating point. At top of atmosphere ($\tau = 0$):

$$L(0, \mu > 0) = e^{-\tau_0/\mu} B(T_0) + \int_0^{\tau_0} e^{-\tau'/\mu} B(T) \frac{d\tau'}{\mu}$$

First term on right side of equation is surface contribution; integral is atmospheric contribution.

Isothermal Layer

Layer of total optical thickness $\Delta\tau$ (= τ_2 - τ_1) at temperature T:

$$L(\tau_1, \mu > 0) = e^{-\Delta\tau/\mu} L(\tau_2, \mu > 0) + B(T) \int_0^{\Delta\tau} e^{-\tau'/\mu} \frac{d\tau'}{\mu} = e^{-\Delta\tau/\mu} L(\tau_2, \mu > 0) + B(T) \left[1 - e^{-\Delta\tau/\mu} \right]$$

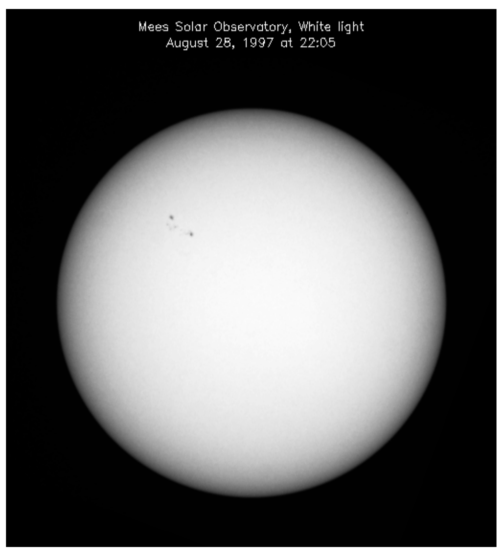
Transmissivity, t, is $e^{-\Delta\tau/\mu}$, ***absorptivity, a,*** is $1-e^{-\Delta\tau/\mu}$ so ***emissivity, ε,*** of layer is $1-e^{-\Delta\tau/\mu}$: $L(\tau_1,\mu>0)=tL(\tau_2,\mu>0)+B(T)[1-t]=tL(\tau_2,\mu>0)+\varepsilon B(T)$

Example:

The 10.14 μm optical depth of the mid-latitude summer atmosphere is 0.27. Most absorption in the Earth's atmospheric window is due to water vapor near the surface. Assume the atmosphere emits as a layer at 285 K and the surface is a blackbody with temperature of 295 K. Calculate the upwelling radiance at 0° and 60°.

Solution (work thos out for yourself):
0°: $L_{10.14\mu m}$ = 8.78 W m^{-2} µm^{-1} sr^{-1} Brightness temperature, T_B = 292.7 K
60°: $L_{10.14\mu m}$ = 8.53 W m^{-2} µm^{-1} sr^{-1} Brightness temperature, T_B = 291 K

This example shows the decrease in radiance for more oblique viewing angles.
This ***limb darkening*** is due to more of the emission for a slant path originating from the colder atmosphere, or, in general, from higher colder layers in an atmosphere.



Solar disk in white light showing limb darkening towards the edges.