1. Problem: The emissivity of quartz at 9.5 µm is about 0.5. Calculate the emitted irradiance at 9.5 µm of quartz at 230 K. Then calculate the brightness temperature.

Emitted irradiance: \( F_\lambda = \epsilon P_\lambda(T) \)

Planck (irradiance) function: \( P_\lambda(T) = \frac{c_1}{\lambda^5 [\exp(c_2/\lambda T)-1]} \)

\( c_1 = 3.517 \times 10^8 \text{ W m}^{-2} \text{ µm}^4 \) and \( c_2 = 1.4388 \times 10^4 \text{ K µm} \) so

\( P_\lambda(T) = 6.7 \text{ W m}^{-2} \text{ µm}^{-1} \) and \( F_\lambda(T) = 3.35 \text{ W m}^{-2} \text{ µm}^{-1} \) and

Brightness temperature, \( T_B = \frac{c_2}{\ln(1+c_1/F_\lambda \lambda^5)} = 208.1 \text{ K} \)

Note that this is much larger than half the physical temperature. **For what wavelengths would the brightness temperature for an emissivity of 0.5 actually be half the physical temperature?**

Recall for Rayleigh-Jeans (long wavelength) limit, \( P_\lambda(T) \propto T \) so \( 0.5P_\lambda(T) \propto 0.5T \). For \( T = 230 \text{ K} \), this occurs in the microwave.

2. Problem: What is the temperature of a blackbody with an emission spectrum that peaks at the same wavelength as the solar spectrum (outside Earth’s atmosphere)?

Wien’s Law: \( T_B = \frac{2898}{\lambda_{\text{max}} (\mu m)} \text{ K} = 6037 \text{ K} \) for \( \lambda_{\text{max}} = 0.48 \mu m \).

Suppose that the sun were replaced by a blackbody of the same radius and distance to Earth. What is the temperature of this blackbody such that the solar irradiance is its presently accepted value (about 1361 W m\(^{-2}\))? The angular width of the sun is about 0.5°.

\[ \sigma T_B^4 = F_0 \left( \frac{R_{S-E}}{R_S} \right)^2 \]

where \( R_{S-E} \) and \( R_S \) are the Sun-Earth distance and radius of the Sun, respectively. Angular width of Sun is 0.5° so \( \tan(0.25\pi/180) = R_S / R_{S-E} \approx 0.25\pi/180 \) so

\[ T_B = 5959 \text{ K} \].
3. Problem The wavelength at which the Planck function has a maximum (for a given temperature) is not an absolute quantity but depends on the independent variable chosen, which is arbitrary. However, there is a peak wavelength that can be defined unambiguously as a median wavelength: the wavelength that divides the Planck function into two regions of equal area. First, convince yourself that this median wavelength is independent of the variable in the Planck function and why. Then determine the displacement law for the median wavelength. What is the median wavelength for a 6000 K blackbody?

Normalize the integral of the Planck function:

\[
\frac{1}{\sigma T^4} \int_0^\infty P_\lambda d\lambda = 1
\]

Making the substitution \( x = \frac{hc}{\lambda K T} \) this becomes

\[
\frac{2 \pi k_B^4}{\sigma c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx
\]

Note that \( \frac{2 \pi k_B^4}{\sigma c^2 h^3} = 0.154 \) so we can define a normalized Planck distribution, \( f(x) \):

\[
f(x) = 0.154 \frac{x^3}{e^x - 1}
\]

Normalized Planck distribution.    Cumulative Planck distribution.

We want to find the value for \( x \) such that \( \int_0^x f(x) \, dx = 0.5 \). We can solve this cumulative distribution numerically; it is plotted in the rightmost figure, above.

\[ P_{cum} = 0.5 \text{ at } x = \frac{hc}{\lambda kT} = 3.5 \]

Median displacement law: \( \lambda_{med} T = \frac{hc}{3.5 k} = 4100 \mu m K \)

for \( T = 6000 K \), \( \lambda_{med} = 0.68 \mu m \)

Compare to Wien’s displacement : \( \lambda_{max} T = 2898 \mu m K \)

For additional reading: Heald, Where is the “Wien peak”? *Amer. J. Phys.*, 71, 1322-23, 2003 (posted on class website)
4. Problem: What fraction of electricity bills for lighting with incandescent lamps is wasted because they emit invisible as well as visible radiation? Take the brightness temperature of incandescent lamps to be 2500 K and assume constant emissivity of the filament. 

\[ f = \frac{\int_{\lambda_1}^{\lambda_2} P_\lambda d\lambda}{\sigma T^4} \]

Find fraction of total irradiance that is in visible portion of spectrum:

This is simply the difference in cumulative distributions that we have already done:

\[ f = p_{\text{cum}}(x_1) - p_{\text{cum}}(x_2) \]

\[ x = \frac{hc}{\lambda kT} \]

Take \( \lambda_1 = 400 \text{ nm} \) and \( \lambda_2 = 700 \text{ nm} \) to be the range of the visible spectrum, then:

\[ x_1 = 14.4 \text{ and } x_2 = 8.2. \]

See figure from previous page: \( f = p_{\text{cum}}(14.4) - p_{\text{cum}}(8.2) = 1 - 0.97 = 0.03 \)

97% is lost!

Emissivity and the Atmospheric Greenhouse Effect

Consider a simple slab model of the atmosphere. The top slab, at temperature \( T_a \), is transparent to shortwave (solar) radiation. The lower slab, at temperature \( T_e \), absorbed all infrared radiation. The two slabs are in radiative equilibrium.

A simple slab model

Balance Equations:

1. Entire system:
   \[ S = \epsilon \sigma T_a^4 + (1-\epsilon) \sigma T_e^4 \]
2. At top slab:
   \[ 2 \epsilon \sigma T_a^4 = \alpha \sigma T_e^4 = \epsilon \sigma T_e^4 \]

\( \epsilon = \alpha \leq 1 \)

\( t = 1 - \epsilon \)

\( \epsilon = \alpha = 1 \)
Solve the balance equations for $T_e^4$:

$$T_e^4 = S/\sigma(1-\varepsilon/2) = 2S/\sigma(1+\delta)$$ since transmissivity $t = 1-\varepsilon$

What is $S$?

If $S_0$ is the total solar irradiance at the top of the atmosphere, the total power intercepted by the Earth is $S_04\pi R_e^2$ where $R_e^2$ is the radius of Earth. Since the total surface area is $4\pi R_e^2$ then $S_0/4$ is the global average. A fraction $a$ (albedo) is reflected to space so we define $S$ as the global averaged net irradiance, (global averaged incident minus reflected):

$$S = S_0(1-a)/4 = 1361 \cdot 0.7 \cdot 0.25 \text{ W m}^{-2} = 238 \text{ W m}^{-2}$$

Adopting for $S_0$ 1361 W m$^{-2}$ and 0.3 for $a$.

Our balance equation from the previous page, solving for the surface temperature, is

$$T_e = \left\{S/\sigma(1-\varepsilon/2)\right\}^{1/4}$$

For $\varepsilon = 1$, $T_e = 303$ K; $\varepsilon = 0$, $T_e = 255$ K; $\varepsilon = 0.8$, $T_e = 289$ K

What is the emitted downward irradiance, $F_↓$, from top slab?

$$F_↓ = \varepsilon\sigma T_a^4 = (\varepsilon\sigma/2)T_e^4 = \varepsilon S/(2-\varepsilon)$$

What is the total (shortwave and longwave) absorbed irradiance at the surface?

$$F_↓ + S = S + \varepsilon S/(2-\varepsilon) = S/(1-\varepsilon/2)$$

Thus, the total absorbed lies between $S(\varepsilon=0)$ and $2S(\varepsilon=1)$. So does this represent the limit of the atmospheric greenhouse?

What if instead we modeled the atmosphere by a set of $N$ slabs?

Downward radiation to the bottom slab:

$$F_↓ = NS$$
Problem: How many atmospheric layers are needed to reach the current average surface temperature of Venus, \( T_0 = 735 \) K? The solar irradiance measured at the top of Earth’s atmosphere is 1361 W m\(^{-2}\); Venus is 0.72 AU from the Sun and its albedo is 0.77.

Bullock and Grinspoon . (*Icarus* 150, 19-37, 2001) hypothesize that during epochs of enhanced outgassing, when large amounts of both H\(_2\)O and SO\(_2\) were released, the albedo of Venus increased due to thickening sulfuric acid clouds. This offset greenhouse warming from increased abundances of these infrared active gases. Warming by as much as 100 K occurred after SO\(_2\) concentrations (and H\(_2\)SO\(_4\) cloud thickness) returned to pre-outgassing levels, but infrared opacity remained enhanced due to H\(_2\)O.

Assuming that the present-day albedo is representative of the inter-outgassing periods, what albedo would be necessary to offset an enhanced greenhouse warming of 100 K?

What increase in cloud optical thickness would produce this increase in albedo? Assume that the clouds do not absorb solar radiation and that the cloud particle scattering properties remain unchanged.

Problem: A paper in the 20 October 2006 issue of *Science* suggests that the projected anthropogenic warming due to increasing concentrations of greenhouse gases could be mitigated, in part, by injecting sulfate aerosol particles into the stratosphere “to provide a negative forcing of the climate system ...” Essentially this is equivalent to increasing the amount of solar radiation reflected to space and decreasing the amount transmitted to the troposphere and surface. The paper states “… that a sustained stratospheric forcing of \(-3\) W m\(^{-2}\) would be sufficient to offset much of the anthropogenic warming expected over the next century.”

Assuming that the change in shortwave radiation due to the injection of stratospheric particles occurs outside of the two slab system, what is the change in radiative equilibrium temperature at the surface (bottom slab)? Assume a value of 1361 W m\(^{-2}\) for the “solar constant” (\(S_0\)) and a shortwave albedo (\(a\)) of 0.3 for the bottom slab, and a current radiative equilibrium temperature (\(T_e\)) of 288 K.
**Extinction Law**

The reduction in radiance over pathlength $ds$ is proportional to the incident radiance:

$$dI_\lambda = -\beta_\lambda I_\lambda ds$$

$\beta_\lambda$: extinction coefficient [length$^{-1}$]  
$\beta_\lambda = \kappa_\lambda \rho$

$\kappa_\lambda$ : mass extinction coefficient [length$^2$ mass$^{-1}$ ]  
$\beta_\lambda = N \sigma_\lambda$

$\sigma_\lambda$: extinction cross section [length$^2$]; $N$ is number density

$$dI_\lambda = -\beta_\lambda I_\lambda ds = -I_\lambda d\tau_\lambda$$

$\tau_\lambda$: optical path; dimensionless depth of layer, measure of strength and number of optically active path of particles along a beam.

$$\tau_\lambda = \int \beta_\lambda ds = \int \kappa_\lambda \rho ds = \int N \sigma_\lambda ds$$

Extinction ($e$) = scattering ($s$) + absorption ($a$):

$$\beta_e = \beta_s + \beta_a$$

$$\kappa_e = \kappa_s + \kappa_a$$

$$\sigma_e = \sigma_s + \sigma_a$$

$$\tau_e = \tau_s + \tau_a$$

*Optical depth* is optical path in vertical from top down:

$$\tau(z) = \int z \kappa(z) \rho(z) dz; \tau(s) = \tau(z) / \mu$$

**Absorber (or scatterer) Amount**

If the mass extinction coefficient $\kappa_\lambda$ is uniform, then the optical depth is $\tau_a = \kappa \rho u$

$u$ is the integrated *absorber amount* between heights $z_1$ and $z_2$ (e.g. g/cm$^2$):

$$u = \int_{z_1}^{z_2} \rho_a(z) dz = \frac{1}{\rho_{a \text{air}}} \int q_a dp$$

$q_a$ is the mass mixing ratio.

For well mixed gases ($q_a$ constant) the absorber amount is proportional to pressure difference across layer.
For more than one absorbing (or scattering) gas (or particle), the optical paths add:

\[ \tau_\lambda = \sum \kappa_{i,\lambda} u_i \]

**Beer-Lambert Law**

Extinction Law (equation of transfer with no sources):

\[ dI_\lambda = -\beta_\lambda I_\lambda ds = -I_\lambda d\tau_\lambda \]

Integrate: \( I_\lambda(s) = I_\lambda(0) \exp(-\tau_\lambda(s)) \)

**Langley Plot**

Consider the direct solar irradiance incident from the direction \( \mu_0 = \cos(\theta_0) \):

\[ F_\lambda = F_{0,\lambda} \exp(-\tau_\lambda/\mu_0) \]

\[ \ln F_\lambda = \ln F_{0,\lambda} - \tau_\lambda(1/\mu_0) = \ln F_{0,\lambda} - m \tau_\lambda \]

Define **airmass**, \( m = 1/\mu_0 = \sec(\theta_0) \)

Plot \( \ln F_\lambda \) vs \( m \): slope \( \tau_\lambda \), intercept \( \ln F_{0,\lambda} \)

Used for calibration; measurement of \( \tau_\lambda \)

**Problem:**

*A sunphotometer has a silicon detector that is highly linear in measuring the direct solar flux. When the solar zenith angle is 60°, the sunphotometer measures a voltage, \( V \), of 1.673 in the 870 nm channel. When calibrated with the Langley plot method, the voltage extrapolated to the top of the atmosphere is \( V_0 = 2.127 \). The effect of molecular absorption is negligible at this wavelength, but the molecular scattering optical depth is 0.015.*

\( a) \) Calculate the aerosol optical depth at this wavelength.
\( b) \) Lidar backscattering measurements show that most of the aerosols are in the 1.5 km thick boundary layer. What is the average volume extinction coefficient?
\( c) \) A typical mass extinction coefficient for sulfate aerosols at this wavelength is \( \kappa = 2.5 \text{ m}^2 \text{ g}^{-1} \). What is an estimate for the aerosol loading (g/m\(^2\))?

**Light Scattering by Particles**

- Scattering is the redirection of light: reflection, refraction, diffraction, etc., are all forms of scattering.
Radiation can be attenuated by absorption or scattering or both. Amount of scattering by a distribution of particles is usually quantified with the volume scattering coefficient:

\[ \beta_{sca}, \text{ the fraction of radiation scattered per distance beam travels} \]

- Particles attenuate by both absorption and scattering:
  \[ \beta_{ext} = \beta_{sca} + \beta_{abs} \]
- The relative amount of scattering to scattering plus absorption (extinction) is given by the single scattering albedo:
  \[ \omega_o = \beta_{sca}/(\beta_{abs} + \beta_{sca}) = \beta_{sca}/\beta_{ext} \]
- Pure scattering: \( \omega_o = 1 \) (conservative scattering)
- Pure absorption: \( \omega_o = 0 \)

**Phase Functions**
- The directional distribution of scattered radiation is the phase function: \( P(\Theta) \) is the fraction scattered in direction \( \Theta \).
- \( \Theta \) is the scattering angle: \( \Theta = 0^\circ \) is forward scattering, \( \Theta = 180^\circ \) is backscattering.
- Isotropic scattering (equal in all directions): \( P(\Theta) = 1 \)

**Normalization**

\[ \int_0^{2\pi} \int_{-1}^{1} P(\Theta) d(\cos \Theta) d\varphi = 4\pi \]

**Asymmetry parameter**

Asymmetry parameter is the first moment (average cosine) of phase function:

\[ g = \frac{1}{2} \int_{-1}^{1} P(\cos \Theta) \cos \Theta d(\cos \Theta) \]

- Equal forward and backward \( g = 0 \)
- Totally forward scattering \( g = 1 \)
- Totally backward scattering \( g = -1 \)
- For water clouds in visible, \( g \sim 0.85 \)

**Scattering Cross Section**

Extinction, scattering, and absorption for single particles is measured in cross sectional area: \( C_{ext} = C_{sca} + C_{abs} \)

Volume extinction coefficient related to cross section by \( \beta_{ext} = N C_{ext} \) (\( N \) is number density)
Scattering and absorption efficiency: unitless ratio of cross section to projected area of particle. For sphere:

\[ Q_{sca} = \frac{C_{sca}}{\pi r^2} \quad Q_{abs} = \frac{C_{abs}}{\pi r^2} \quad Q_{ext} = \frac{C_{ext}}{\pi r^2} \]

**Aerosols Particles** (aerosol refers to particles plus carrier gas):

- Small particles (< 10 μm), solid or liquid
- Sulfates (sulfuric acid, ammonium sulfate), hydrated depending on RH. From gas-to-particle conversion of SO₂, DMS.
- Mineral -wind blown dust from soils (quartz, iron oxide) (large size).
- Sea salt - from air bubbles bursting at sea surface (large).
- Combustion - from fossil fuels, forest burning (small).

- Aitken nuclei (nucleation mode) < 0.1 μm
- Large aerosol (accumulation mode) 0.1 to 2 μm (most important for radiation)
- Giant aerosol (coarse mode) > 2 μm

**Clouds**

Types:
- Cumulonimbus: deep thunderstorms
- Cirrus: high altitude (cold) ice clouds
- Altostratus, altocumulus: midlevel
- Stratus and cumulus: low (boundary layer) (usually) liquid clouds.

**Cloud Microphysics**

- Water droplets
  - radius: 2 - 40 μm radius (typical: 10 μm)
  - Concentration: 50 - 500 cm⁻³
  - Liquid water content: 0.03 - 3.0 g m⁻³

- Ice clouds - non-spherical (hexagonal) particles
  - Size: length 10 - 1000 μm (typical 100 μm)
  - Concentration: 1 - 1000 liter⁻¹
  - Ice water content: 0.001 - 0.1 g m⁻³
Particle Size Distributions

- Size distribution $n(r)$ is the number of particles per volume per radius interval.
- Typical units: cm$^{-3}$ μm$^{-1}$
- Total number of particles per volume: $N = \int n(r)dr$
- Mass per volume of air or liquid water content (LWC) is: $LWC = \frac{4\pi}{3} \sigma \int \frac{r^3}{n(r)}dr$

Scattering from a Distribution of Particles

- Integrate over particle size distributions and multiple particle types.

Extinction (or optical depth):

$$\beta_{ext} = \int_{0}^{\infty} C_{ext} n(r)dr = \int_{0}^{\infty} \pi r^2 Q_{ext} n(r)dr$$

Single scattering albedo: $\sigma_o = \beta_{sca} / \beta_{ext}$