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Punctuated Hamiltonian Models of Structured Turbulence

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Abstract. High Reynolds number turbulence is often dominated by coherent structures. A new class of dynamical system, called punctuated Hamiltonian dynamics, appears particularly appropriate for modeling structured turbulence. Punctuated Hamiltonian models successfully capture the quantitative behavior of several classes of two-dimensional turbulence and appear promising for large-scale geophysical turbulence.

1. Introduction

It is now well established that high Reynolds number turbulence can be populated with coherent vortex structures whose evolution dominates the flow. In large-scale geophysical turbulence, such as flows in the Earth's atmosphere and oceans and on the giant planets, environmental anisotropy due to rotation and stratification play important roles in determining the dynamics [**13**]. In anisotropic turbulence coherent vortices occur at large scales and are associated with the inverse cascade of energy from small scales to large scales. Some examples of large scale coherent vortices are atmospheric vortices such as hurricanes and cyclones, oceanic vortices such as Gulf Stream Rings, and the Great Red Spot of Jupiter.

Due to the broad range of scales present in high Reynolds number turbulence one cannot capture all the degrees of freedom in a numerical simulation. Thus, we would like to model turbulence in terms of only the most important degrees of freedom. The concentration of vorticity into coherent structures suggests that they are the important degrees of freedom and that the remainder of the flow may be safely ignored. This strategy leads to a class of models, called punctuated Hamiltonian models, which are the subject of this paper. These models use the coherent structures as their elemental basis and describe the evolution of the turbulence in terms of the population of coherent structures. Punctuated Hamiltonian models have been successfully used to describe several different classes of two-dimensional

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turbulence [**2, 7, 16, 19**] and appear very promising for other types of structured turbulence.

As the anisotropy of a flow increases, one passes through a number of flow regimes. Environmental anisotropy due to rotation can be characterized by the Rossby number, $R = V/fL$ where V and L are typical velocities and horizontal length scales, and f is the Coriolis parameter, $f = 2\Omega \sin\phi$, with Ω being the planetary rotation frequency and ϕ being latitude. Stable stratification is characterized by the Froude number $F = V/NH$ where H is a typical vertical length scale and N is the Brunt-Väisälä frequency, i.e., the oscillation frequency of fluid parcels displaced vertically from their equilibrium position. Note that both nondimensional numbers can be thought of as the ratio of a frequency associated with advection and a frequency associated with the environmental anisotropy; thus the anisotropy is strong when these numbers are small. The asymptotic regime associated with fast rotation $R \ll 1$, and comparably strong stable stratification $F \ll 1$, and $R \sim F$ is called quasigeostrophy (QG). If, however, the stratification is much stronger than the rotation, $F^2 \ll R^2 \ll 1$ then the flow is, to lowest order, two-dimensional (2D). Since 2D flow is simpler than three-dimensional flow it has received a lot of attention and is often used as a starting point for theories and computations of geophysical turbulence. In this paper we focus on 2D turbulence.

2. Numerical Simulations of Two-Dimensional Decaying Turbulence

The equations of motion for 2D turbulence are

$$\zeta_t + J(\psi, \zeta) = D + F, \tag{1}$$

where ζ is the vorticity, ψ is the streamfunction, $J(A, B) = A_x B_y - A_y B_x$, and D and F are dissipation and forcing, respectively. The vorticity and streamfunction are related to the velocity $\vec{u}$ by $\vec{u} = (-\psi_x, \psi_y)$ and $\zeta = (\nabla \times \vec{u}) \cdot \hat{z}$. In numerical simulations hyperviscous dissipation is often used, $D = (-1)^{p-1}\nu_p \nabla^{2p}\zeta$, where ν_p is the coefficient of hyperviscosity; $p = 1$ is molecular viscosity. In this section we consider decaying turbulence, $F = 0$.

We discuss numerical simulations of 2D decaying turbulence which fall into one of two categories based on their initial condition. These initial conditions are given by an ensemble of Fourier modes with random phases whose amplitudes are determined by either a narrow-band [**11**] or broad-band [**3, 4, 10**] energy spectrum (Figure 1). In the case of a narrow-band initial condition, the energy has a well defined characteristic scale: the wavenumber of the spectrum peak. This characteristic scale, which is chosen to be significantly smaller than the domain scale, is inherited by the initial coherent vortices. In this narrow-band case the evolution of the vortex population is characterized by temporal self-similarity. On the other hand, the broad-band initial spectrum does not have a characteristic scale, resulting in vortices which do not have a characteristic scale. Here the population is characterized by spatial self-similarity. This dependence on initial conditions, which was studied by Santangelo, et.al. [**17**], indicates that 2D turbulence is not universal. There are a number of open questions surrounding this non-universality: Would 2D turbulence display universal behavior if only one waited long enough, making the apparent non-universality a finite-time effect? If not, does 2D turbulence have of a

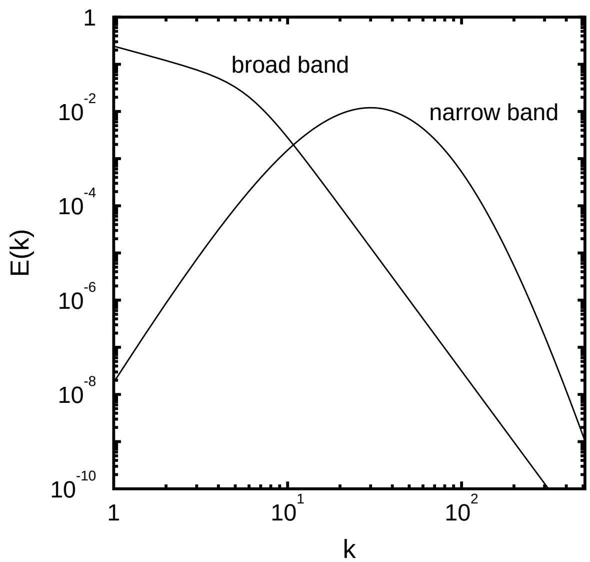

FIGURE 1. Energy spectra for narrow-band and broad-band initial conditions.

small number of well-defined universality classes, within which any initial condition evolves to a single vortex population?

In both classes of initial conditions the evolution shares the following characteristics. Random initial conditions self-organize into coherent vortices on a time-scale roughly equal to a vortex turnover time. Subsequently the vortices move through the fluid due to their mutual advection. When the vortices are well separated, the evolution is conservative. When two vortices approach closely they deform each other and dissipation can occur. If the two vortices are of the same sign and sufficiently close, they merge to form a single, larger vortex. Other dissipative interactions include the destruction of a vortex if it gets close to a larger vortex of either sign due to the strong shear generated by the larger vortex. The peak value of vorticity within the vortices is roughly independent of size and does not significantly change in time. The vortex population is thus described by a probability distribution function for vortex size, $\rho(r,t)$

The evolution in the narrow-band case is described by a vortex-based scaling theory [**7, 19**]. The theory assumes that $\rho(r,t)$ has well-defined moments and evolves self-similarly in time:

$$\rho(r,t)\,dr = p(x)\,dx, \qquad x = \frac{r}{r_{avg}(t)}, \qquad r_{avg}(t) = \int r\,\rho(r,t)\,dr.$$

Thus, ρ is completely determined by the function p and the time evolution of the average vortex size. Additional assumptions in the scaling theory are that the number of vortices evolves as $N \sim t^{-\xi}$, the energy is conserved, and the peak vorticity inside the vortices is constant. The numerical simulations indicate that these assumptions are all satisfied and $\xi \approx 0.70 - 0.75$. The scaling theory then predicts the time evolution of all moments of ρ in terms of ξ. In particular it predicts the evolution of the average vortex size r_{avg} and average absolute value vortex circulation Γ_{avg} as well as field quantities such as the enstrophy Z and vorticity kurtosis K by virtue of their being dominated by the vorticity in the vortices:

$$r_{avg} \sim t^{\xi/4}, \qquad \Gamma_{avg} \sim t^{\xi/2}, \qquad Z \sim t^{-\xi/2}, \qquad K \sim t^{\xi/2}.$$

In the scaling regime of broad-band decaying turbulence the distribution of vortex properties is constant in time, except for the edge effects due to finite resolution and domain size [**3, 4**]. Individual vortices merge and grow resulting in a flux of vortices moving through a constant distribution. It is found from numerical simulations that $\rho(r) \sim r^{-\alpha}$ with $\alpha \approx 1.9$. The average vortex size for this distribution is completely determined by the large and small scale cutoffs; thus the moments are not well-defined and the above mentioned temporal scaling theory does not apply.

3. Punctuated Hamiltonian Models of Two-Dimensional Decaying Turbulence

The importance of coherent vortices in decaying turbulence leads one to consider constructing models where the vortices are fundamental entities. This perspective requires separation of the flow into the vortices and the background. The separate dynamics of the vortices and background, as well as their interaction, are then represented by a simplified dynamical model. These representations can be chosen with varying levels of detail, resulting in a hierarchy of models [**12**]. Here we discuss the simplest model in this hierarchy, which was independently proposed by Carnevale, et.al. [**7**] for the narrow band case, and Benzi, et.al. [**2**] for the broad-band case.

The first simplification is that we assume that all the important vorticity is in the coherent vortices themselves. Thus, any dynamics associated with vorticity filaments is ignored. Further, the filaments are assumed to irreversibly cascade to smaller and smaller scales until they are dissipated along with their associated enstrophy. The background flow thus acts purely as a sink, absorbing properties cast off by the vortices. In the case of forced turbulence, which will be considered later, this assumption requires modification.

A single coherent vortex, labeled by index i, has its vorticity ζ_i localized in a small region and the distribution of vorticity within that region represents its internal degrees of freedom. However, when one is far away from the vortex the effects of these internal degrees of freedom become negligible and all that matters is its position, $\vec{x}_i$, and circulation, Γ_i. Thus, a distant vortex is well represented by a point vortex:

$$\zeta_i(\vec{x}) = \Gamma_i \delta(\vec{x} - \vec{x}_i).$$

When all vortices are far apart, the system is assumed to evolve as a population of point vortices. Such a population constitutes a Hamiltonian dynamical system [**1**]. Each vortex contributes one degree of freedom so a reasonably sized population of a hundred vortices has a very high dimensional phase space. For more than a few vortices the system is nonintegrable.[1] The Hamiltonian is given by

$$H = \sum_{i,j,i\neq j} \Gamma_i \Gamma_j h(\vec{x}_i, \vec{x}_j),$$

the equations of motion, which are noncanonical due to the extra factor of Γ_i, are

$$\Gamma_i \dot{x}_i = \frac{\partial H}{\partial y_i} \qquad \Gamma_i \dot{y}_i = -\frac{\partial H}{\partial x_i},$$

and h depends on the boundary conditions. For an infinite plane h takes the simple form

$$h(\vec{x}_i, \vec{x}_j) = -\frac{1}{4\pi} \ln |\vec{x}_i - \vec{x}_j| \, .$$

For the periodic boundary conditions used in the numerical simulations of 2D turbulence h is more complex due to the contribution of image vortices created by the boundaries [**6, 15, 18**]. The models described here use periodic boundary conditions.

Vortex merger is fundamentally a dissipative process involving the vortices' internal degrees and cannot be captured by the Hamiltonian point vortex system. Thus to model the evolution of the vortex population the point vortex representation needs to modified. Vortex merger occurs when two same-sign vortices approach closely, where close is defined relative to their size. Since point vortices have no size, there is no way to decide whether two point vortices are close in this regard. Thus, the first modification is to assign a radius, r_i, to each vortex. When vortices are far apart this size is irrelevant and the vortices evolve according to chaotic Hamiltonian point vortex dynamics. However when two vortices approach close enough then dissipative processes must be included.

The simplest model for dissipative processes assumes that the dominant dissipative process is the merger of same-sign vortices. In particular, we ignore the possibility of more complex outcomes from same-sign close approach and ignore entirely dissipative processes arising from the close approach of opposite sign vortices. When vortices merge they rather quickly shed vorticity filaments to the background which ultimately cascade to small scales and are dissipated. In this approximation once two vortices become close enough for merger to be initiated the process is irreversible even if its ultimate conclusion takes some time. Thus we parameterize merger as an instantaneous event in which two close same-sign vortices merge to form a single final vortex and ignore all other dissipative process. It remains to decide how close vortices must be to initiate merger. While this critical merger distance d_c does depend on the vortex size, the evolution of the population appears to be relatively insensitive to the details of how d_c is computed and one can use a simple rule derived from numerical simulations [**19**],

$$d_c(r_i, r_j) = 3.3 \frac{r_i + r_j}{2}.$$

It remains to specify how the two vortices merge to form a larger vortex. Using the conserved properties of the temporal scaling theory, energy and peak vorticity,

[1]The exact number depends on the boundary conditions which determine the number of independent invariants [**1, 18**].

one finds that the new vortex radius r_k and circulation Γ_k are simple functions of the two original radii r_i, r_j, and circulations Γ_i, Γ_j:

$$r_k^4 = r_i^4 + r_j^4, \qquad \Gamma_k^2 = \Gamma_i^2 + \Gamma_j^2.$$

Thus, the simplest vortex based model of decaying turbulence consists of chaotic Hamiltonian evolution of a population of point vortices where each vortex is described by a position, circulation, and size. Whenever two vortices with the same-sign circulation approach closer than the critical merger distance the Hamiltonian dynamics is stopped, the two vortices are replaced with a single merged vortex, and the Hamiltonian dynamics continues with one fewer vortex. This class of model, where Hamiltonian dynamics is interrupted by instantaneous dissipative events, is called a *punctuated Hamiltonian model* and represents a new class of dynamical system.

In high Reynolds number turbulence one could naively expect that since the coefficient of the dissipation approaches zero the dissipation becomes unimportant. However, what actually happens is that dissipation remains important but becomes increasingly intermittent. This is precisely the situation captured by punctuated Hamiltonian models, which, unlike traditional dissipative systems, are conservative for the majority of their evolution (indeed, they are only dissipative at isolated points in time). In a population of well-separated vortices, dissipative events are rare and while they are instantaneously a singular perturbation to the Hamiltonian dynamics, in a time average sense they are a very weak perturbation.

Comparisons of this simplest punctuated Hamiltonian model with numerical simulations show that it works surprisingly well. In the narrow-band case the vortex population indeed evolves self-similarly (Figure 2), and the value of ξ, $\xi = 0.72$, matches the simulations [**19**]. In the broad-band case the vortex population evolves to a distribution with $\alpha = 2$, again matching the appropriate numerical simulation [**2**]. Thus, in two separate regimes of decaying 2D turbulence the very simplest punctuated Hamiltonian model quantitatively captures the turbulent behavior.

A first step in constructing a mathematical theory of punctuated Hamiltonian dynamics is to consider the phase space behavior of trajectories. Punctuated Hamiltonian trajectories evolve in a $2N$ dimensional phase space corresponding to the N vortices and are restricted to the region of phase space with no close same-sign vortices. The excluded region is bounded by a collection of hyper-surfaces where each hyper-surface is defined by the critical merger distance for a single same-sign pair. Whenever a trajectory touches one of these hyper-surfaces the system jumps to a new phase space with two fewer dimensions. At the point of contact the original phase space may be broken up into a $2N - 4$ dimensional subspace with no same-sign close approaches and a 4 dimensional subspace containing the two close same-sign vortices.[2] After merger the trajectory evolves in a new $2N - 2$ dimensional phase space consisting of the same $2N - 4$ dimensional subspace with no close same-sign vortices and a new 2 dimensional subspace containing the merged vortex. The evolution of the vortex population is governed by the rate of mergers

[2]It is possible that three or more same-sign vortices will approach closely at the same time, i.e., the trajectory touches one hyper-surface exactly at its intersection with one or more other hyper-surfaces. This possibility, however, is extremely unlikely and is ignored here.

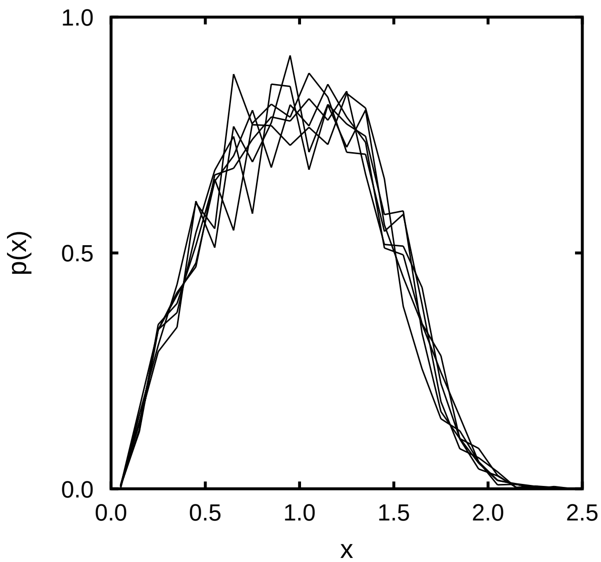

FIGURE 2. Distribution of number of vortices p vs. scaled vortex size x at several times for a punctuated Hamiltonian model of narrow-band decaying turbulence. Temporal self-similarity is indicated by time-invariance of $p(x)$.

and by the distribution of mergers among different sized vortices. In the the context of phase space trajectories this becomes a first-passage-time problem: Given a high dimensional chaotic Hamiltonian system with a trajectory outside a set of hyper-surfaces, what is the distribution of times for a trajectory to first intersect one of the hyper-surfaces, and what is the distribution among the hyper-surfaces of that first intersection?

Punctuated Hamiltonian models for decaying turbulence have a trivial long-time attractor: a single opposite-sign vortex pair. This pair propagates steadily corresponding to an attractor which is a fixed point in an appropriate comoving frame. Decaying turbulence on the other hand has an ultimate end state of no motion. The above attractor corresponds to the vortex pair achieved at the end of the nonlinear evolution of decaying turbulence. In decaying turbulence this vortex pair subsequently decays to the state of rest on a diffusive timescale which goes to infinity with the Reynolds number. This illustrates that punctuated Hamiltonian dynamics is dissipative in the same sense as infinite Reynolds number turbulence, allowing dissipative vortex interactions but no diffusive decay of vorticity. The more interesting dynamical state, however, is not the ultimate fixed point but rather the transient self-similar regime which arises when there are a large number of vortices.

Overall, the simplest punctuated Hamiltonian models work extremely well for modeling the behavior seen in current simulations of 2D decaying turbulence. These simulations however, which typically use spectral methods, can only reach moderate Reynolds numbers and questions remain about very high Reynolds numbers. Simulations using contour surgery indicate that dissipative processes ignored in

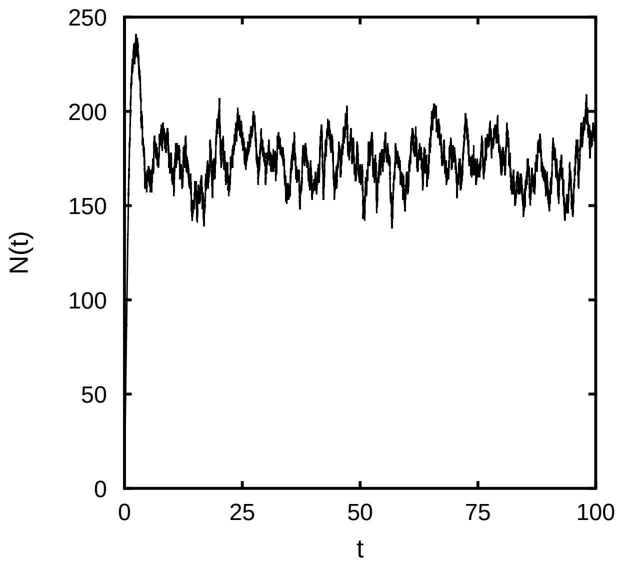

FIGURE 3. Vortex number vs. time in a punctuated Hamiltonian model of stationary turbulence.

this simplest parameterization could become important as the Reynolds number approaches infinity [**8**]. However, contour surgery is itself a parameterized dissipation and it is not clear how well contour surgery truly represents infinite Reynolds number. Recently, Riccardi, et.al. [**16**] constructed a model of dissipative processes based on simple contour surgery simulations of two merging vortices which includes the possibility of obtaining one, two, or three vortices following a dissipative interaction. With this more complicated punctuated Hamiltonian model, Riccardi, et.al., were able to capture the behavior seen in the contour surgery turbulence simulations. The overall conclusion seems to be that punctuated Hamiltonian models can capture a wide range of behavior, and, in particular, can successfully model the dynamics of a given turbulence simulation provided the dissipative processes are modeled to match those in the simulation. However, the question of which dissipative processes are the most important as the Reynolds number goes to infinity is still unanswered.

4. Two-Dimensional Stationary Turbulence.

The addition of forcing to the equations of motion (1) allows the flow to reach a nontrivial statistical equilibrium. Forced two-dimensional turbulence seems to have a variety of behaviors depending on the details of the forcing F [**5, 9**]. We would expect that if punctuated Hamiltonian models can be used for stationary turbulence they would be most appropriate for a regime where vortices are dominant. Thus we focus on the regime studied by Borue [**5**] where energy is supplied at small scales and at an appropriate intensity for small scale vortices to form. Here we describe some preliminary results from a punctuated Hamiltonian model for stationary turbulence.

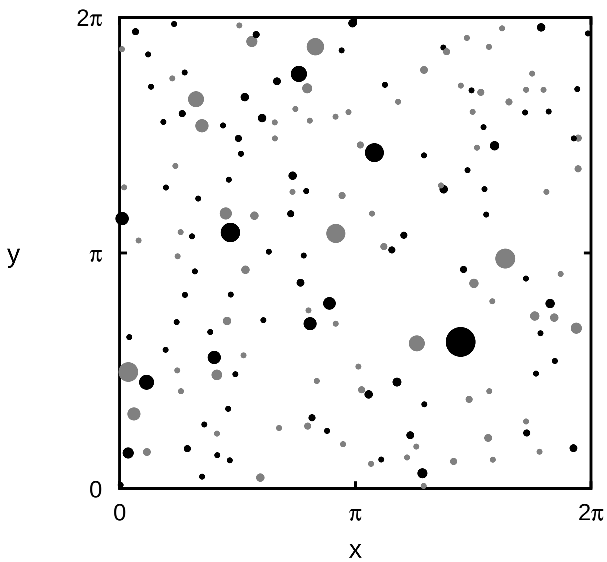

FIGURE 4. Spatial configuration of vortices at one instant in time in a punctuated Hamiltonian model of stationary turbulence.

Once small-scale vortices emerge from the forcing, they evolve via the same mutual interactions present in the decaying case. To avoid the build-up of energy at the domain scale of the computation, Borue uses, in addition to the standard small scale hyperviscosity, a large-scale dissipation operator $-\nu\nabla^{-16}\zeta$. The result of the simulation is that the steady-state vortex population has a distribution of vortices with size $\rho(r) \sim r^{-4}$.

In order to model the effects of forcing and large-scale dissipation we add two new punctuation rules to the model. Forcing is represented by placing a pair of small vortices with size $r = r_{min}$ at random positions in the fluid with frequency ω. Thus, unlike in decaying turbulence, the background in forced turbulence is active and the injection of energy creates vortices at the forcing scale. Large-scale dissipation is represented by removing any vortex which gets larger than some maximum size r_{max}. The particular simulation presented here has $r_{max} = 0.2$ which is significantly smaller than the domain scale 2π, and $r_{min} = 0.04$, resulting in a factor of 5 in vortex size. The vorticity is chosen so that the smallest vortices have a turnover time (defined here as the induced speed at the vortex edge) of unity. The vortex creation frequency $\omega = 200$ results in a rapid injection of energy and enstrophy.

The model is integrated from an initial condition consisting of four vortices with $r = r_{min}$. As seen in Figure 3, the number of vortices rapidly grows and reaches a statistical equilibrium. A typical vortex configuration appears in Figure 4. The average vortex size, Figure 5, is also stationary but shows larger fluctuations than the vortex number. The time averaged vortex size distribution, Figure 6, agrees surprisingly well with the r^{-4} slope found by Borue. However there are significant deviations from this scaling behavior at both ends of the distribution.

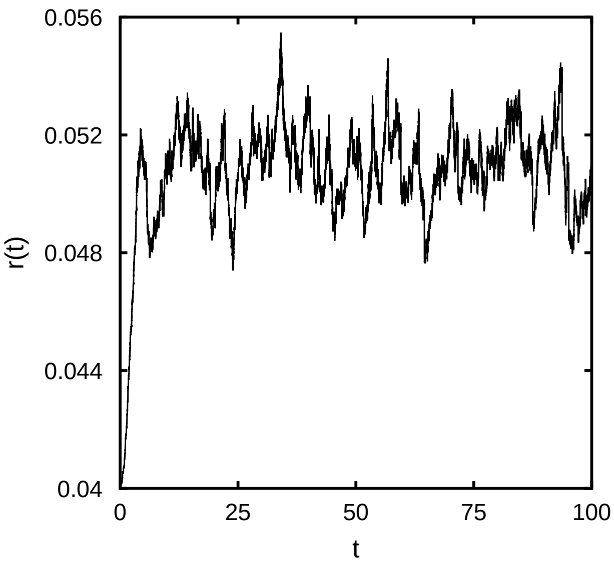

FIGURE 5. Average vortex radius vs. time in a punctuated Hamiltonian model of stationary turbulence.

Thus, punctuated Hamiltonian dynamics works extremely well in describing the behavior of stationary two-dimensional turbulence in the regime where the flow is dominated by coherent vortices. More studies are needed with a wider range between r_{min} and r_{max} to better see the scaling regime. Additionally, it would be interesting to study how the equilibrium state is affected by the forcing rate ω.

Unlike the models for decaying turbulence, the model for stationary turbulence has a nontrivial attractor. The trajectory wanders through phase spaces of varying dimensions but has well defined average properties. The instantaneous phase space dimension can never be less than four, corresponding to a single opposite sign-pair which is the fixed point attractor of the decaying model. If such a state were achieved (which is not the case in the numerical simulations), then the dimension of the system would increase due to the injection of new vortices. The instantaneous phase space dimension is bounded from above by the fact that one can only pack a certain number of finite sized vortices into the bounded domain before they are close enough to merge and decrease the dimension of phase space. In practice the system stays well away from these limits and in the simulation presented here the phase space dimension remains around $300 - 400$. The novel mixture of dissipative and Hamiltonian features in punctuated Hamiltonian dynamics is reflected in the fact that the attractor is composed of Hamiltonian phase spaces of different dimensions, each of which has no attractor of its own. It is the jumping between these Hamiltonian spaces, the punctuation, that provides the dissipation needed to achieve an attractor. It may be that the best way to talk about this attractor is to embed the system into the highest dimensional phase space possible, and then the punctuation is manifested by the trajectory expanding or contracting into larger and smaller subspaces.

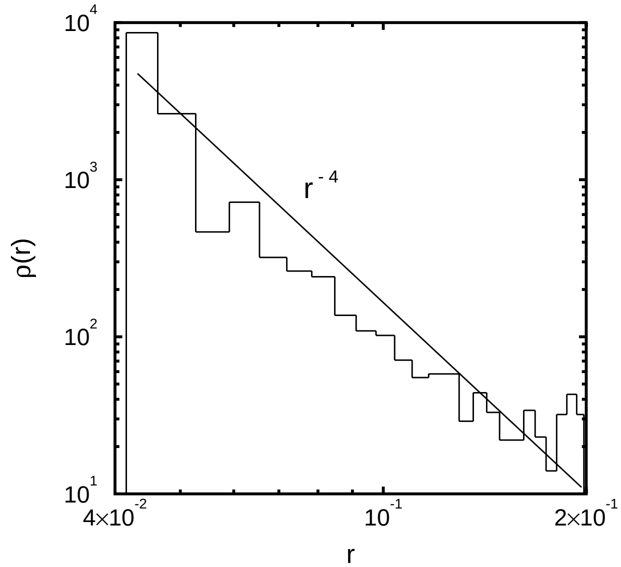

FIGURE 6. Distribution of vortex size in a punctuated Hamiltonian model of stationary turbulence.

5. Discussion

Structured turbulence appears in systems other than 2D flow. For example, large-scale 3D atmospheric and oceanographic flows are highly anisotropic and are described by the QG equations. Numerical simulations of decaying QG turbulence show the emergence of 3D coherent vortices which evolve through horizontal merger, similar to that in 2D turbulence, and through vertical alignment [**13, 14**]. Given the similarities between 3D QG and 2D turbulence it is natural to think that punctuated Hamiltonian dynamics may be an appropriate description of large-scale atmospheric and oceanographic turbulence. However, while the study of 2D vortices goes back over a hundred years, the study of QG vortices is much more recent and much work is needed to determine the appropriate punctuation rules.

Several regimes of 2D turbulence have now been successfully modeled by punctuated Hamiltonian dynamical systems. Punctuated Hamiltonian dynamics provides a novel class of dynamical system with very different properties from traditional Hamiltonian or dissipative systems. Further, this class of dynamical system appears ideally suited for modeling high Reynolds number structured turbulence. The mathematics of punctuated Hamiltonian systems have not been studied as yet, and there is the possibility that significant results may be obtained. Furthermore, any mathematical results on punctuated Hamiltonian dynamics will most likely have an immediate impact on our understanding of structured turbulence.

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