Vortex Statistics from Eulerian and Lagrangian Time Series

Claudia Pasquero and Antonello Provenzale
ISAC-CNR, Corso Fiume 4, 10133 Torino, Italy

Jeffrey B. Weiss
PAOS, University of Colorado, Boulder, Colorado 80309-0311
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Coherent vortices are an important component of the dynamics of geophysical turbulence, but direct estimates of the properties of the vortex population from measured data is usually difficult. Motivated by this problem, we propose a new method for determining the statistical properties of coherent vortices in two-dimensional turbulence based on a small number of Lagrangian and Eulerian time series. The method provides reliable estimates of the mean vortex size and vortex number density.

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Introduction.—Turbulent geophysical flows are characterized by the presence of long-lived coherent vortices that significantly affect the dynamics [1–3]. Typically, coherent vortices live for times that are hundreds or thousands of times their rotation period.

Although these vortices usually occupy only a small fraction of space, they concentrate a large portion of the energy and vorticity of the flow and can significantly affect tracer transport [4,5]. For this reason, the statistics of the vortex population, such as the average vortex number, size, and energy, are crucial ingredients in scaling theories of structured turbulence [6,7] and in parametrizing the effects of mesoscale turbulence in ocean circulation and climate models [8].

When full knowledge of the velocity field is available, as in high-resolution numerical simulations, the statistical properties of the vortex population can be determined by adopting specific vortex-census algorithms [9]. In Nature, however, the determination of vortex statistics can be difficult, due to the fact that only partial information is available. Typically, data sets consist of Lagrangian velocity and position time series, furnished by freely moving ocean floats or atmospheric balloons, and Eulerian velocity time series, recorded by chains of oceanic current meters or fixed-point measurements in the atmosphere.

In this Letter, we address the problem of reconstructing vortex statistics from single-point Eulerian or Lagrangian times series. We describe a new method that successfully estimates vortex statistics in homogeneous two-dimensional turbulence. Two-dimensional turbulence approximately describes mesoscale geophysical turbulence and has been widely studied in the past. In particular, two-dimensional turbulence is characterized by the spontaneous emergence of long-lived coherent vortices [1,2]. The field then becomes a two-phase system, composed of high-energy coherent vortices, and a quasirandom, low-energy, low-vorticity background turbulent field [10].

Two-dimensional turbulence.—Two-dimensional turbulence is described by the equation $\frac{\partial \omega}{\partial t} + [\psi, \omega] = f + d$, where ψ is the stream function, $\omega = \nabla^2 \psi$ is vorticity, $[\cdot, \cdot]$ is the Jacobian operator, and f and d represent forcing and dissipation, respectively. The fluid velocity is $\mathbf{u} = (u_x, u_y) = (\partial_y \psi, -\partial_x \psi)$. Here, we numerically integrate this equation by a pseudospectral method in a square domain with size L. Dissipation is parametrized by a hyperviscous term acting on small scales, and by an inverse Laplacian acting on the scales of the simulation domain. We force by keeping the energy spectrum fixed at a selected wave number, k_F. Forcing and dissipation are kept as low as possible in order to reach the highest Reynolds number compatible with the numerical resolution. The flow evolves from random initial conditions to a statistically stationary state, where the average number, energy, and circulation of the vortices display small fluctuations of less than 5% of their average value. Further details on these simulations are given in [4,8].

In the following, we discuss the results obtained from simulations forced at intermediate scales, $k_F = 40$. After an initial transient, the vorticity field becomes punctuated with a large number of coherent vortices that self-organize from the turbulent background. With this type of forcing, the size of the vortices is usually smaller than the forcing scale. Figure 1 shows a snapshot of the vorticity field.

Even when the full vorticity field is available, the identification of the vortices is a delicate issue: Several criteria have been proposed in the literature, based on topological characteristics [9,11–13] or on the local properties of the flow [14]. Here, we use the criterion based on the Okubo-Weiss parameter [15,16] to extract the vortex statistics from the full vorticity field. The Okubo-Weiss parameter is defined as $Q = S^2 - \omega^2$, where $S^2 = (\partial_x u_x - \partial_y u_y)^2 + (\partial_y u_x + \partial_x u_y)^2$. The vortex cores are characterized by strongly negative values of Q. In the following, we identify the vortex cores as those circular regions with negative Okubo-Weiss parameter around a

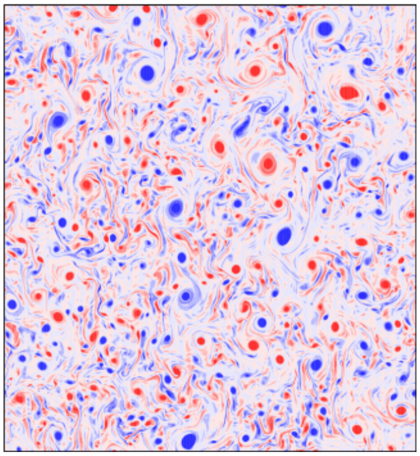

FIG. 1 (color online). A snapshot of the vorticity field.

peak of vorticity. This census indicates that 120 ± 5 vortices are simultaneously present in the field, whose mean size is $(0.014 \pm 0.004)L$. The characteristics of the field inside and outside the vortices differ noticeably: The mean enstrophy inside vortices, Z_v, is about 1000 times the mean enstrophy outside vortices. A measure of the rotation period of the vortices is given by $T_v = Z_v^{-1/2}$. In the following, we use T_v as our nondimensional time unit.

Eulerian and Lagrangian statistics.—The presence of coherent vortices induces important differences in the statistical properties of Eulerian and Lagrangian velocity time series. One crucial fact is that Lagrangian tracers rarely cross the edge of coherent structures. They usually enter or leave a vortex only during vortex filamentation, merging, or during the birth of a new vortex. As a consequence, Lagrangian particles can be trapped in vortices for long periods of time [4].

In our work, Eulerian data result from measurements at fixed sites in the simulation domain, while Lagrangian data result from passively advected tracers in the time-evolving turbulent flow. The Lagrangian velocity of the particles is given by the Eulerian velocity at the particle location. The particle trajectories are numerically integrated with a leapfrog scheme. A third-order cubic spline interpolation of the Eulerian velocity field is used to compute the velocity at the particle positions. The length of the Eulerian and Lagrangian time series is $T = 1700T_v$, and 4096 Eulerian measurement sites and Lagrangian particles are used.

The velocity autocorrelation function is defined as

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\overline{\mathbf{u}(t) \cdot \mathbf{u}(t+\tau)}}{\sigma^2} dt,$$

where σ^2 is the velocity variance, and the overline indicates average over the ensemble of the time series. Depending on the type of data, we obtain either a Lagrangian or a Eulerian autocorrelation function, denoted by $R_L(\tau)$ or $R_E(\tau)$, respectively. The autocorrelation functions are displayed in Fig. 2(a). The Lagrangian integral time, $T_L = \lim_{T\to\infty} \int_0^T R_L(\tau)d\tau$, is $T_L = 17.8T_v$, while the analogous Eulerian integral time is $T_E = 50T_v$. Note that, due to the finite length of the time series, these integral times have been computed over the total length of the simulation, T. The Lagrangian autocorrelation function decays faster than the Eulerian one, a fact that we attribute to the fast rotation of particles close to, or inside, vortices. A particle which moves in a circle has a (vectorial) velocity that decorrelates on a time scale related to the rotational velocity and therefore depends directly on the vortex strength. As discussed in [8], the Lagrangian autocorrelation function at short times is dominated by the presence of the rapidly rotating coherent structures. Once the vortex-induced component of the Lagrangian velocity is decorrelated, only the background-induced component is left, which leads to a decay of R_L with a time scale of roughly T_E.

At larger times, the Lagrangian autocorrelation $R_L(\tau)$ indicates almost complete decorrelation of the Lagrangian velocities, while $R_E(\tau)$ displays noticeable oscillations. These oscillations are due to the vacillation between high-energy vortices which intermittently pass by the Eulerian measurement site and the low-energy background turbulence. Since the kinetic energy of the vortices is much larger than that of the background, the vortex contribution dominates over that of the background, and induces long-term correlations and anticorrelations associated with the passage of same-sign or opposite-sign vortices. In contrast, Lagrangian particles tend to stay either outside or inside vortices for long times. The alternation between the two turbulent phases seen in the Eulerian signals is then lost in the Lagrangian trajectories. As a consequence, the average Lagrangian autocorrelation does not display long-term oscillations. This difference is reflected in the shape of the power spectra: At low frequency, Eulerian data display larger variance than Lagrangian data, while the opposite is true at high frequency, where the contribution due to particle spinning inside vortices becomes significant. This leads to steeper slopes of Eulerian spectra compared to Lagrangian spectra, as shown in Fig. 2(b).

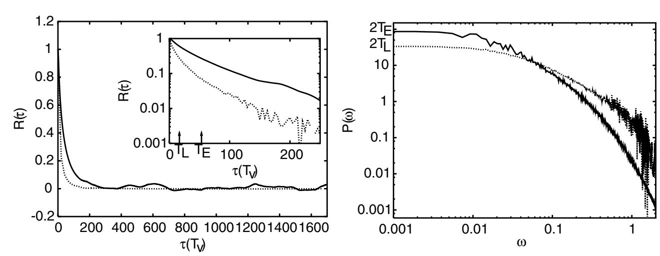

FIG. 2. (a) Velocity autocorrelation function. (b) Power spectrum. Dotted lines refer to Lagrangian data and solid lines to Eulerian data.

The impermeability of vortex cores to inward and outward particle fluxes has important consequences for the determination of the velocity distributions. When sufficient data are available, Eulerian and Lagrangian velocity distributions converge, and, in the case of 2D turbulence at high Reynolds number, become non-Gaussian [10]. In the case considered here, the kurtosis of the velocity distribution is $K = 4.5$. However, the convergence of the velocity distribution estimated from an ensemble of finite measurements is different for Eulerian and Lagrangian data. To investigate this, we randomly selected m Eulerian or Lagrangian time series out of the available set, and computed the variance and the kurtosis of the velocity distribution obtained from the m time series. In Fig. 3, we show the estimates of the variance and kurtosis, as a function of m, obtained from an ensemble of 100 different random choices of m time series out of the entire set. The vertical bars include 95% of the results obtained from each individual group of m time series. The variability in the ensemble of realizations is smaller for the Eulerian data, indicating that the one-point velocity statistics are estimated more accurately by fixed measurement sites than particle trajectories. For the Eulerian data, the estimates are closer to the true values, and even a small number of measurement sites provides good estimates of the moments of the velocity distribution. By contrast, Lagrangian particles do not explore the full turbulent field and more data is required to obtain similar quality estimates of the velocity distribution.

Reconstructing vortex statistics.—We now use the information recorded in the velocity time series to reconstruct the statistics of the vortex population. From the Eulerian time series, we can compute the time that a vortex takes to cross a Eulerian measurement site, τ_v, and the time interval between vortex crossings, τ_b.

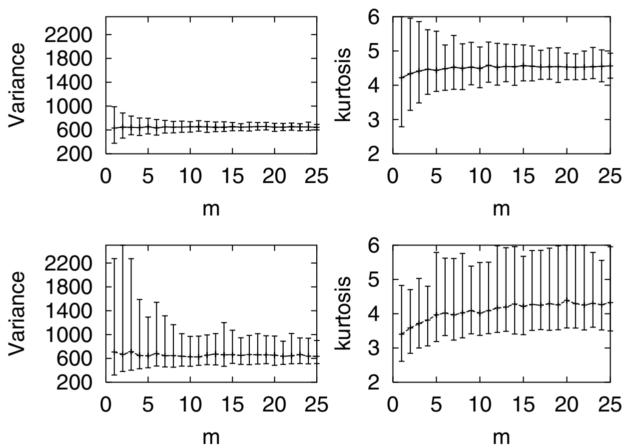

FIG. 3. Velocity variance and kurtosis calculated from an ensemble of 100 randomly extracted subsamples of m Eulerian measurement sites (upper panels) and Lagrangian trajectories (lower panels) from the entire data set. The vertical bars include 95% of the members of each ensemble.

Their average values, obtained from the full set of available time series, are $\overline{\tau_v} = (7.2 \pm 1.0)T_v$, and $\overline{\tau_b} = (260 \pm 50)T_v$. Figure 4 shows the estimates of $\overline{\tau_v}$ and $\overline{\tau_b}$ as a function of the number of vortex crossings n. The vertical bars include 95% of the values obtained from the analysis of the individual vortex transits. Note that with just ten vortex crossings one obtains estimates of $\overline{\tau_v}$ and $\overline{\tau_b}$ that are correct up to a factor of 2 in more than 95% of the cases.

The estimate of the crossing time and of the time between vortex passages requires the identification of vortex edges. To this end, we have used a threshold on the value of the Okubo-Weiss parameter. Note, however, that the estimate of the Okubo-Weiss parameter requires knowledge of the velocity gradient. When this quantity is not available, one can identify vortex passages using the velocity only [17]: The edges of vortices are associated with peaks in the kinetic energy, while the center is characterized by small kinetic energy. However, in this case the results will be less precise.

Given that $\overline{\tau_v}$ is much shorter than the Eulerian decorrelation time T_E, the motion of a vortex on a time scale $\overline{\tau_v}$ can be considered ballistic. The size of a vortex is thus given by $R = u_a \tau_v$, where u_a is the advection speed of the vortex. Since the average velocity of a vortex in a system of many vortices is not significantly different from that of a particle in the background [18], the advection speed u_a turns out to be statistically independent of vortex size. This inference is confirmed by the fact that the correlation coefficient between vortex radius and velocity, obtained from 4000 independent couples of values, is 0.002. The mean size is thus $\overline{R} = \overline{\tau_v}/\overline{u_a^{-1}}$. The value of $\overline{u_a^{-1}}$ can be estimated from the speed of Lagrangian tracers outside vortices, or, if the statistical sample is large enough, by the velocity recorded at Eulerian measurement sites during the intervals between vortex crossings. In the present simulation, the two methods give $\overline{u_a^{-1}} = (570 \pm 5)T_v/L$. This gives an estimate of the average vortex size, $\overline{R} = (0.013 \pm 0.002)L$, which is in excellent agreement with the estimate provided by applying the vortex-census to the entire vorticity field.

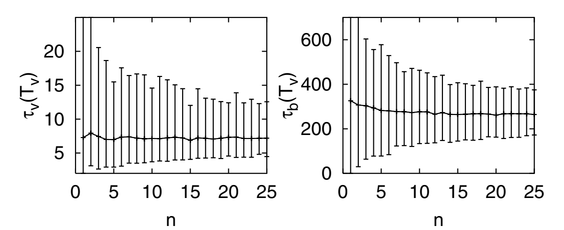

FIG. 4. Average time that a vortex takes to cross a Eulerian measurement site (left panel), and average time interval between vortex crossings (right panel), as a function of the number of vortex crossings n included in the calculation. For each value of n, an ensemble of 400 members is used. The vertical bars include 95% of the members of each ensemble.

One could use similar arguments to obtain equations for higher moments of the vortex size distribution. However, these moments depend on higher moments of u_a^{-1}, which depend sensitively on small u_a, and cannot be reliably estimated.

The number of coherent vortices can also be inferred from an analysis of the velocity time series. The area swept out by a vortex during a time interval Δt is a strip of width R and length $\int_0^{\Delta t} u_a(t)dt$. For time intervals longer than the decorrelation time T_L, memory of the initial velocity is lost, and the mean length of a path becomes $\overline{u_a}\Delta t$. Since u_a and R are independent, the mean area covered by a vortex in the time interval Δt is then $\overline{R}\,\overline{u_a}\Delta t$. In a time interval $t^* = L^2/(\overline{R}\,\overline{u_a})$, a vortex covers, on average, an area equal to the entire simulation domain. Thus, a Eulerian measurement point is visited by a given vortex once every t^*. In the presence of N vortices, the average time interval between the passage of two vortices past a given measurement site is $\overline{\tau_b} + \overline{\tau_v} = t^*/N$. Using the above estimate of $\overline{R}$, it follows that the number of vortices is given by

$$N = \frac{L^2 \overline{u_a^{-1}}}{(\overline{\tau_b} + \overline{\tau_v})\overline{\tau_v u_a}}.$$

The Eulerian data then gives $N = 103 \pm 25$, in good agreement with the estimate of the average number of vortices provided by the vortex census.

We have applied the same approach to another simulation of two-dimensional turbulence, forced at a different scale, and have similarly obtained good estimates of the number and average size of the vortices.

Conclusions.—We have discussed the differences between Eulerian and Lagrangian measurements of two-dimensional turbulence. We have proposed a new method for determining the elementary statistics of vortex populations in geophysical turbulence and verified the method in the case of two-dimensional turbulence. The method requires a small number of Eulerian time series and allows for reliably estimating the number density and average size of the coherent vortices present in the field. When available, velocity Lagrangian time series can be used to get a good estimate of the advection speed of the vortices. The method assumes the existence of coherent vortices and uses knowledge of their general properties. We suggest that this type of approach can be extended to estimate the statistical properties of the coherent vortex populations in the ocean and the atmosphere. In realistic geophysical flows, however, the existence of large-scale inhomogeneities in the distribution of eddy kinetic energy and the presence of dynamical structures such as jets and planetary waves can greatly complicate the picture. This method should thus be applied only to relatively homogeneous, vortex-dominated portions of the flow, and after the removal of temporal trends. Future work may allow extension of this method to more complex situations.

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