Rossby Waves in a two-layer shallow water system

In class we saw that the two-layer linearized system can be written in terms of two uncoupled equations, one for a barotropic streamfunction and one for a baroclinic streamfunction. Each equation has its own Rossby wave, where the barotropic Rossby wave has a dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2},$$

(1)

and the baroclinic Rossby wave has the dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2 + k_d^2},$$

(2)

where the internal deformation radius is $L_d^I = 1/k_d^I$ with $k_d^I = \sqrt{F_1 + F_2}$.

Here, you will instead find the waves in the two layer system, and show that two waves correspond to the same barotropic and baroclinic waves, getting the same result by following a different path.

Start with the two-layer shallow water equations for a fluid on a beta-plane with a solid top and bottom:

$$q_1 = \nabla^2 \psi_1 + \beta y + F_1(\psi_2 - \psi_1),$$

$$q_2 = \nabla^2 \psi_2 + \beta y + F_2(\psi_1 - \psi_2),$$

$$\frac{Dq_1}{Dt} = 0,$$

$$\frac{Dq_2}{Dt} = 0,$$

where $F_i = f_0^2/(g'H_i)$. Linearize about a state of rest and find the dispersion relation for Rossby waves of the form

$$\psi_i' = A_i \exp (i(kx + ly - \omega t)).$$

(3)

Show that the two roots correspond to the barotropic and baroclinic modes.