Solutions of the Momentum Equation

In this homework you will explore solutions of the inviscid momentum equation for a fluid with constant density,

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p,$$

where units of mass have been chosen to set the constant density to $\rho = 1$.

1. Find the pressure field consistent with a fluid at rest, $\vec{u} = 0$.

   With $\vec{u} = 0$ the momentum equation becomes $\nabla p = 0$, which requires that the pressure be constant.

2. A parallel shear flow is one where the velocity is constant in the direction of flow, but can change in directions perpendicular to the flow. Consider the parallel shear flow $\vec{u} = U(y) \hat{x}$ where $U(y)$ is an arbitrary scalar function.

   a. Sketch the vector velocity field when $U(y) = \cos(2y)$.

   ![Figure 1: Sketch of the velocity field $\vec{u} = \cos(2y) \hat{x}$](image)

   b. Find the pressure field consistent with $\vec{u} = U(y) \hat{x}$.
Since $\vec{u}$ doesn’t depend on time, $\partial \vec{u}/\partial t = 0$. $(\vec{u} \cdot \nabla) \vec{u} = (U(y)\partial/\partial x) U(y) \hat{x} = 0$.
So again, $\nabla p = 0$ and the pressure is constant.

3. Consider an oscillating velocity field $\vec{u} = \vec{U} \sin(\omega t)$ where $\vec{U} = U \hat{x} + V \hat{y} + W \hat{z}$ and $U$, $V$, and $W$ are constants. Find the pressure field consistent with this velocity.

Since $\vec{u}$ is independent of position, the advective term is zero. Then $\partial \vec{u}/\partial t = \omega \vec{U} \cos(\omega t) = -\nabla p$. In component form:

\[
\frac{\partial p}{\partial x} = -\omega U \cos(\omega t), \\
\frac{\partial p}{\partial y} = -\omega V \cos(\omega t), \\
\frac{\partial p}{\partial z} = -\omega W \cos(\omega t).
\]

The pressure field consistent these equations is $p = -\omega \cos(\omega t) (U \hat{x} + V \hat{y} + W \hat{z})$.

4. Nonrotating updraft. Consider the velocity field $\vec{u} = -x \hat{x} - y \hat{y} + 2z \hat{z}$ and suppose the ground is at $z = 0$.
   a. Show that this flow is incompressible.

   This is a straightforward calculation of the divergence: $\nabla \cdot \vec{u} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(2z) = 0$.

   b. Create a visualization of this flow field that effectively communicates what the velocity looks like.

   Visualizing three dimensional vector fields is difficult. Often the best way is with two dimensional slices.
Figure 2: Velocity field $\vec{u} = -x\hat{x} - y\hat{y} + 2z\hat{z}$.

c. Find the pressure field.

Since the velocity field is constant in time the time derivative is zero. The momentum equation is then

$$(\vec{u} \cdot \nabla)\vec{u} = x\hat{x} + y\hat{y} + 4z\hat{z} = -\nabla p.$$  \hspace{1cm} (1)

The $\hat{x}$-component gives $\partial p/\partial x = -x$ which requires $p = -x^2/2 + f(y, z)$ where $f$ is an unknown function. The $\hat{y}$-component, $\partial p/\partial y = -y$ requires $f(y, z) = -y^2/2 + g(z)$ where $g$ is an unknown function. The $\hat{z}$-component, $\partial p/\partial z = -4z$ then gives $g = -2z^2 + p_0$, where $p_0$ is an arbitrary constant. The pressure is then $p = -x^2/2 - y^2/2 - 2z^2 + p_0$.

5. Rotating updraft. Now consider the momentum equation on a rotating planet,

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \hat{z} \times \vec{u} = -\nabla p,$$

where we have set the Coriolis parameter $f$ to unity, $\vec{f} = \hat{z}$.

a. Show that there is no pressure field consistent with the velocity field from problem 4.

The Coriolis force is $\hat{z} \times \vec{u} = -x\hat{y} + y\hat{x}$ and the momentum equation is now Equation 1 with an additional term from the Coriolis force

$$(\vec{u} \cdot \nabla)\vec{u} = x\hat{x} + y\hat{y} + 4z\hat{z} - x\hat{y} + y\hat{x} = -\nabla p.$$
The $\hat{x}$-component requires $\partial p/\partial x = x + y$, resulting in
\[ p = x^2/2 + xy + f(y, z), \quad (2) \]
with an unknown function $f$. The $\hat{y}$-component requires $\partial p/\partial y = -x + y$, resulting in
\[ p = -xy + y^2/2 + g(x, z), \quad (3) \]
with an unknown function $g$. Neither $f(y, z)$ nor $g(x, z)$ can have a term with both $x$ and $y$. Equation 2 says $p$ must have a term $+xy$, while Equation 3 says it must have a term $-xy$. This is inconsistent. Hence there is no pressure field consistent with the momentum equation.

b. What does it mean that there is no pressure field consistent with the velocity field?

The momentum equation is a statement of Newton’s 2nd law in a fluid. The forces acting on a rotating fluid are the Coriolis force and the pressure gradient force. Part a) shows that there is no possible pressure field that produces a pressure gradient force that satisfies Newton’s 2nd law for this velocity field. Hence, this velocity field is unphysical and cannot be realized in this situation.

It is possible to use the velocity field as an initial condition. The velocity field as given is independent of time. However, since the forces cannot balance with zero time derivative, if this field was used as an initial condition there would be a net acceleration resulting in a change in the velocity field, $\partial \vec{u}/\partial t \neq 0$.

c. Use what you know about updrafts on the Earth to qualitatively describe what the velocity looks like for an updraft on a rotating planet and compare it with the velocity field from problem 4.

For the scales of motion where rotation is important in the atmosphere, updrafts take the form of synoptic scale mid-latitude cyclones. The velocity from 4 has inward radial flow and no rotation. However in mid-latitude cyclones, the velocity has a strong cyclonic rotational component. This rotational velocity allows the Coriolis force to balance the pressure gradient, resulting in a geostrophically balanced flow.