Mathematical Methods in Fluid Dynamics

Scalars and Vectors

Scalar – any quantity which can be fully specified by a single number

Vector – a quantity which requires both a magnitude and direction to be fully specified

What are some examples of scalar and vector quantities?

Vector Notation: $\vec{u} = u_x \vec{i} + u_y \vec{j}$

Magnitude of a vector: $|\vec{u}| = \sqrt{{u_x}^2 + {u_y}^2}$

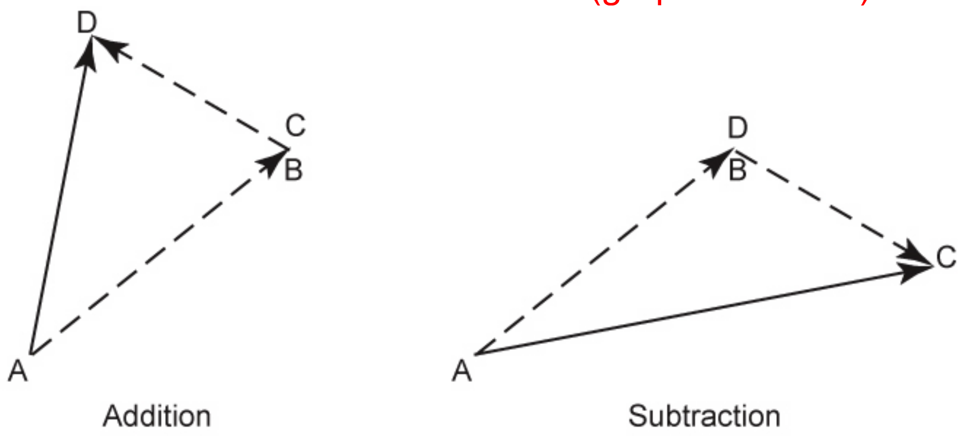
Direction of a vector: $\text{Direction} = \arctan\left(u_y / u_x\right)$

Coordinate systems on the Earth:

For a coordinate system with (x,y,z) we use unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Algebra of Vectors

Addition and subtraction of two vectors (graphic method):



Addition of two vectors $\vec{u}$ and $\vec{v}$:

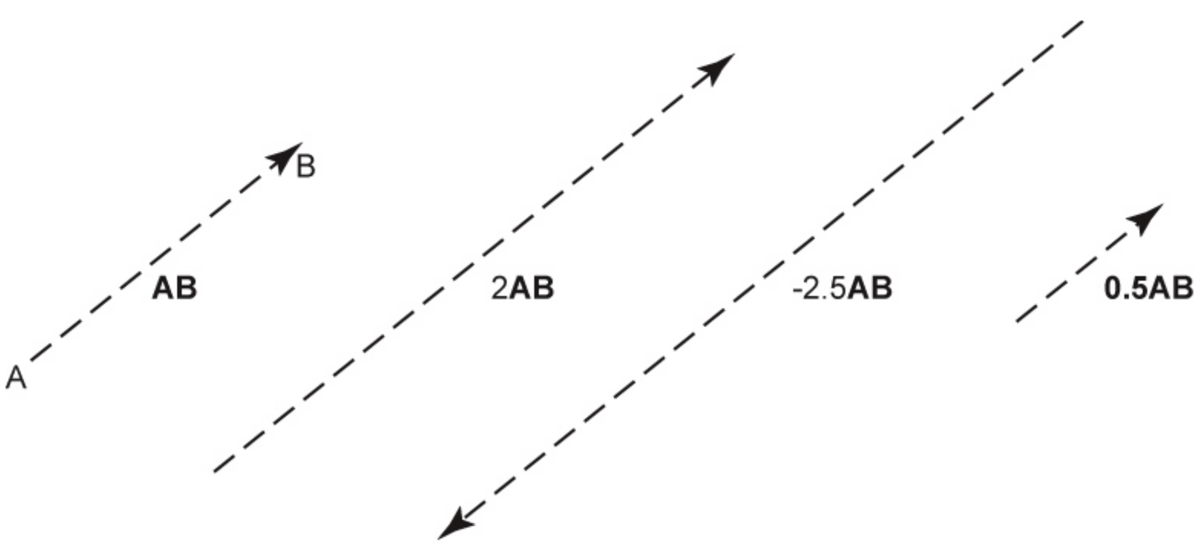
$$\vec{u} = u_x\vec{i} + u_y\vec{j} + u_z\vec{k} \quad and \quad \vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\Rightarrow \vec{u} + \vec{v} = (u_x + v_x)\vec{i} + (u_y + v_y)\vec{j} + (u_z + v_z)\vec{k}$$

Subtraction of two vectors $\vec{u}$ and $\vec{v}$:

$$\vec{u} - \vec{v} = (u_x - v_x)\vec{i} + (u_y - v_y)\vec{j} + (u_z - v_z)\vec{k}$$

Multiplication of a vector by a scalar (graphic method):



Multiplication of vector $\vec{u}$ by scalar *c*:

$$c\vec{u} = cu_x\vec{i} + cu_y\vec{j} + cu_z\vec{k}$$

How does the direction and magnitude of a vector change due to multiplication by a scalar?

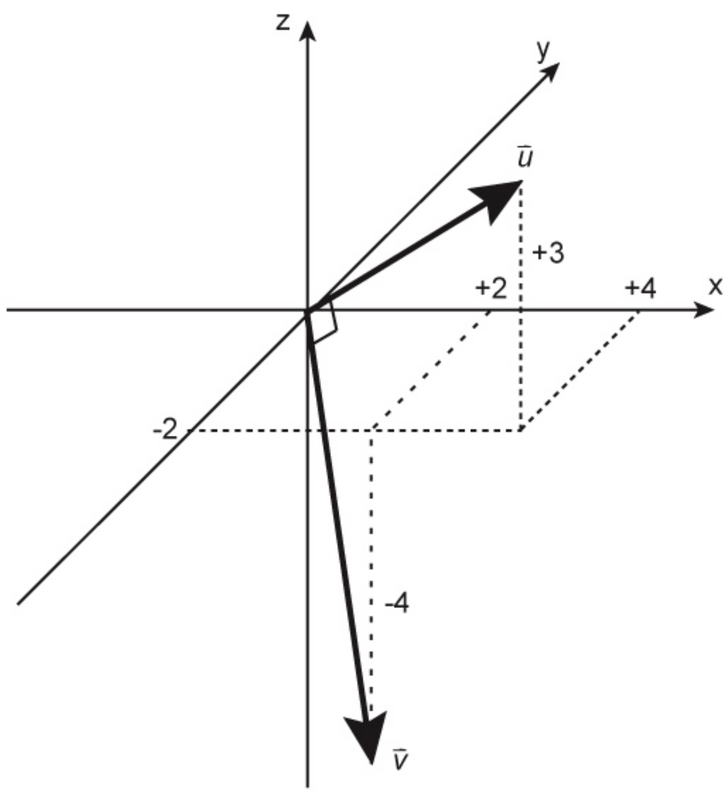
Multiplication of two vectors

Scalar product (or dot product) of $\vec{u}$ and $\vec{v}$:

$$\vec{u} \bullet \vec{v} = u_x v_x + u_y v_y + u_z v_z = |\vec{u}||\vec{v}|\cos\theta$$

When will the dot product of two vectors be equal to zero?

What does this tell us about the direction of the two vectors relative to each other?



Vector product (or cross product) of $\vec{u}$ and $\vec{v}$:

$$\vec{u} \times \vec{v} = (u_y v_z - u_z v_y)\vec{i} + (u_z v_x - u_x v_z)\vec{j} + (u_x v_y - u_y v_x)\vec{k}$$

or

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= (u_y v_z - u_z v_y)\vec{i} + (u_z v_x - u_x v_z)\vec{j} + (u_x v_y - u_y v_x)\vec{k}$$

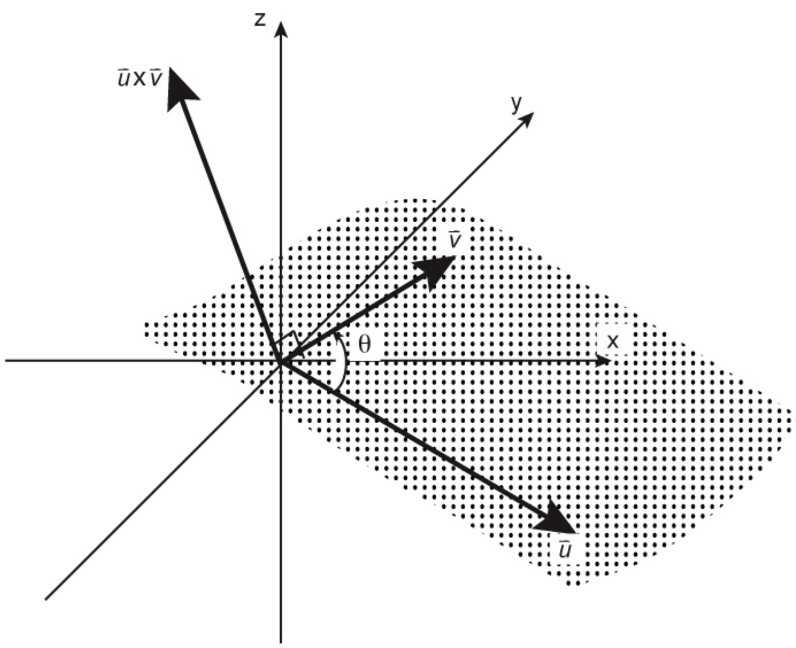
What is the direction of the vector that results from the cross product?

The right hand rule

What is the magnitude of this vector?

Magnitude = $|\vec{u}||\vec{v}|\sin\theta$

When will the cross product be equal to zero?



Scalar and Vector Fields

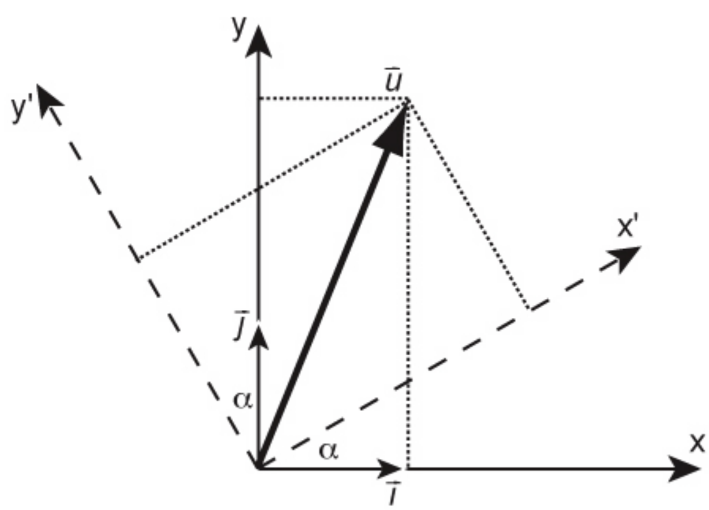
Field – a quantity defined over a given coordinate space
The field is a function of the three coordinates of position and also of time.

$T = f(x,y,z,t)$

T and *p* are examples of scalar fields and wind is an example of a vector field.

Coordinate Systems on the Earth

How do scalar and vector fields change when the coordinate system is rotated?



How would vector $\vec{u} = u_x\vec{i} + u_y\vec{j}$ change under a rotation of the coordinate system?

$$\vec{u}' = (u_x \cos\alpha + u_y \sin\alpha)\vec{i}\,' + (u_y \cos\alpha - u_x \sin\alpha)\vec{j}\,'$$

Meteorologists traditionally define a coordinate system relative to the Earth.

Gradients of Vectors

The vectors we consider in meteorology often vary in space and time (i.e. they are functions of both space and time).

We can show this, for a wind velocity vector, as $\vec{U}(x, y, z, t)$.

This vector, in component form, can be written as:

$$\vec{U}(x, y, z, t) = u(x, y, z, t)\vec{\imath} + v(x, y, z, t)\vec{\jmath} + w(x, y, z, t)\vec{k}$$

Written in this way we see that this vector consists of zonal, meridional, and vertical components of the wind that vary in all three spatial directions (x,y,z) and vary in time.

The variation of the wind vector with respect to any one of the independent variables can be written as a partial derivative.

$$\frac{\partial \vec{U}}{\partial t} = \frac{\partial u}{\partial t}\vec{\imath} + \frac{\partial v}{\partial t}\vec{\jmath} + \frac{\partial w}{\partial t}\vec{k}$$

What does each term in this equation represent physically?

What if we considered the partial derivative of $\vec{U}(x, y, z, t) = u(x, y, z, t)\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$ with respect to x or y?