Circulation and Vorticity

Example: Rotation in the atmosphere – water vapor satellite animation

Circulation – a macroscopic measure of rotation for a finite area of a fluid

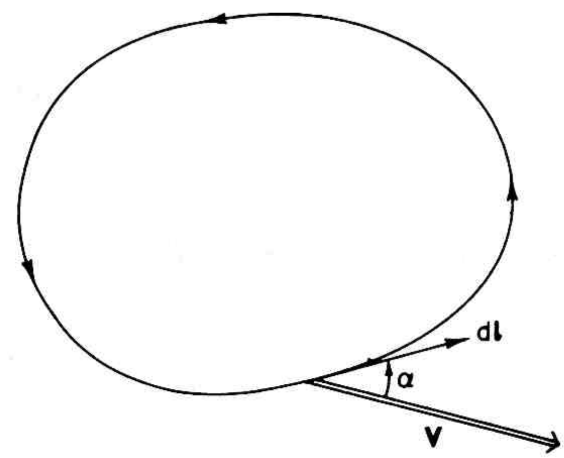
Vorticity – a microscopic measure of rotation at any point in a fluid

Circulation is a scalar quantity while vorticity is a vector quantity.

Circulation Theorem

Circulation is defined as the line integral around a closed contour in a fluid of the velocity component tangent to the contour

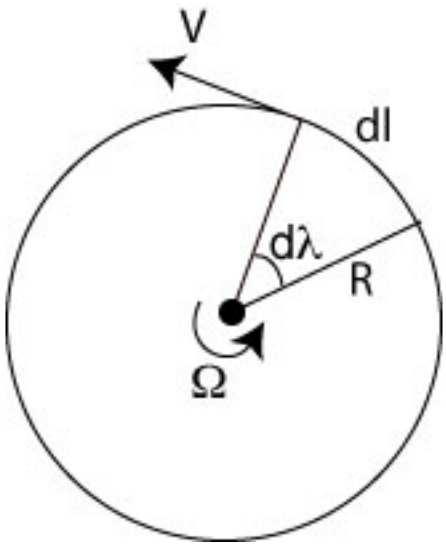
$$C \equiv \oint \vec{U} \cdot d\vec{l} = \oint |\vec{U}| \cos\alpha \, dl$$



By convention *C* is evaluated for counterclockwise integration around the contour.

When will *C* be positive (negative)?

Consider a fluid in solid body rotation with angular velocity Ω.



For a circular ring of fluid with radius R:

$V = \Omega R$

$dl = Rd\lambda$

Then the circulation is:

$$C = \oint \Omega R dl = 2\pi \Omega R^2$$

What is the relationship between C and angular momentum in this case?

What is the relationship between C and angular velocity in this case?

C can be used to describe the rotation in cases where it is difficult to define an axis of rotation and thus angular velocity.

How will C change over time?

Take total derivative of C in an inertial reference frame:

$$\frac{D_a C_a}{Dt} = \frac{D_a}{Dt}\oint \vec{U}_a \cdot d\vec{l} = \oint \frac{D_a \vec{U}_a}{Dt} \cdot d\vec{l} + \oint \vec{U}_a \cdot \frac{D_a d\vec{l}}{Dt}$$

But: $\oint \frac{D_a \vec{U}_a}{Dt} \cdot d\vec{l}$ is just the line integral of the acceleration of wind:

$$\frac{D_a \vec{U}_a}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \Phi,$$

where we have neglected the friction force, do not need to consider the Coriolis force for an inertial reference frame, and have expressed the gravitational force as a gradient of the geopotential.

Noting that: $\frac{d\vec{l}}{Dt} = \vec{U}_a$ then $\frac{D_a d\vec{l}}{Dt} = D_a \vec{U}_a$ and:

$$\oint \vec{U}_a \cdot \frac{D_a d\vec{l}}{Dt} = \oint \vec{U}_a \cdot D_a \vec{U}_a = \frac{1}{2}\oint D_a\left(\vec{U}_a \cdot \vec{U}_a\right) = 0$$

Using this the change in absolute circulation following the motion is:

$$\frac{D_a C_a}{Dt} = \oint -\frac{1}{\rho}\nabla p \cdot d\vec{l} - \oint \nabla\Phi \cdot d\vec{l}$$

The line integral of the gravitational force is given by:

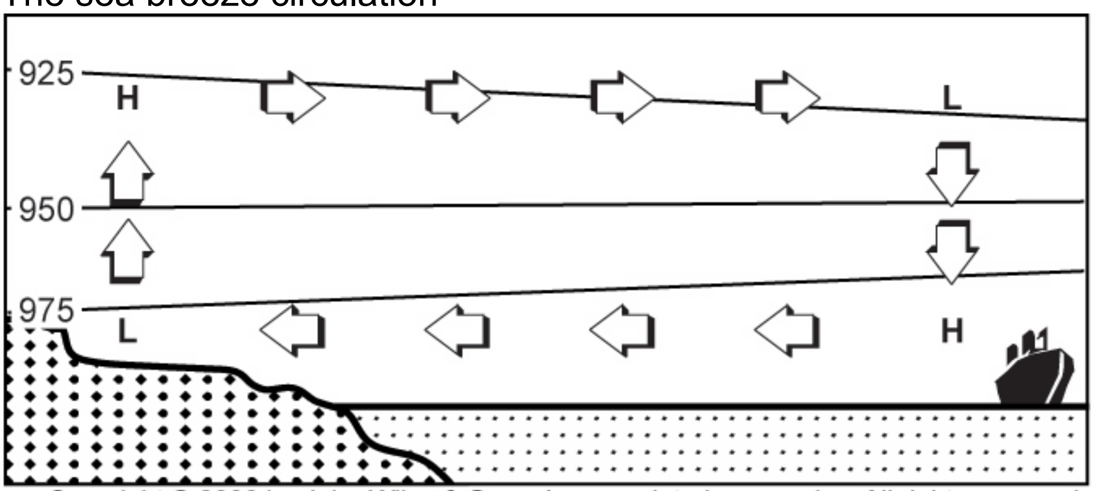
$$-\oint \nabla\Phi \cdot d\vec{l} = \oint -g\vec{k} \cdot d\vec{l} = \oint -g dz = \oint -d\Phi = 0$$

Then the change in circulation is given by:

$$\frac{D_a C_a}{Dt} = \oint -\frac{1}{\rho}\nabla p \cdot d\vec{l} = \oint -\frac{1}{\rho} dp$$

What is the physical interpretation of each term in this equation?

The sea breeze circulation



What are the physical mechanisms that lead to sea breeze formation?

Example: Calculate the change in circulation associated with a sea breeze

For a barotropic fluid $\rho = \rho(p)$ and $\oint -\frac{1}{\rho}dp = 0$ so:

$$\frac{D_a C_a}{Dt} = 0$$

and absolute circulation is conserved following the motion.

This is known as Kelvin's circulation theorem.

What is the relationship between absolute and relative circulation?

C_a – absolute circulation
C_e – circulation due to the rotation of the Earth
C – relative circulation

$C_a = C_e + C$

The circulation due to the rotation of the Earth is given by:

$C_e = \oint \vec{U}_e \cdot d\vec{l}$, where

$U_e = \vec{\Omega} \times \vec{r}$ is the velocity due to the rotation of the Earth

Stokes' theorem:

$$\oint \vec{F} \cdot d\vec{s} = \iint_A \left(\nabla \times \vec{F}\right) \cdot \vec{n} dA,$$

where $\vec{n}$ is the unit vector normal to the area A and the direction of $\vec{n}$ is defined by the right-hand rule for counterclockwise integration of the line integral.

Using Stokes' theorem we can rewrite our equation for C_e as:

$$C_e = \oint \vec{U}_e \cdot d\vec{l} = \iint_A \left(\nabla \times \vec{U}_e\right) \cdot \vec{n} dA$$

Using the vector identity:

$$\left(\nabla\times\vec{U}_e\right)=\nabla\times\left(\vec{\Omega}\times\vec{r}\right)=\nabla\times\left(\vec{\Omega}\times\vec{R}\right)=\Omega\nabla\cdot\vec{R}=2\vec{\Omega}$$

For a horizontal plane $\vec{n}$ is directed in the vertical direction and

$$\left(\nabla\times\vec{U}_e\right)\cdot\vec{n}=2\vec{\Omega}\cdot\vec{n}=2\Omega\sin\phi=f\,,$$

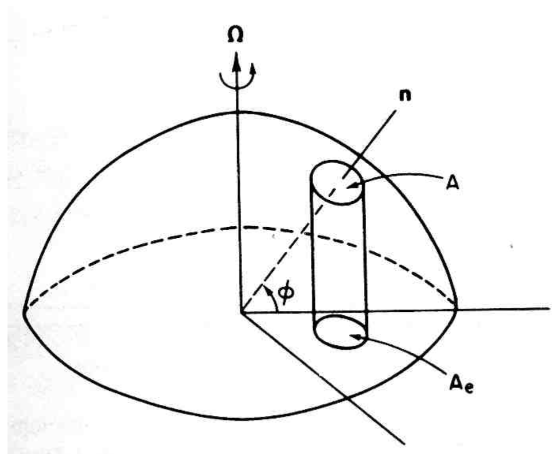
where we have used $\vec{\Omega}=\Omega\cos\phi\vec{j}+\Omega\sin\phi\vec{k}$

Then:

$$C_e=\iint\limits_A\left(\nabla\times\vec{U}_e\right)\cdot\vec{n}\,dA=2\Omega\sin\phi A=2\Omega A_e\,,$$

where A_e is the area projected onto the equatorial plane and:

$A_e = A\sin\phi$



Then:

$C = C_a - C_e = C_a - 2\Omega A\sin\phi = C_a - 2\Omega A_e$

The relative circulation (C) is the difference between the absolute circulation (C_a) and the circulation due to the rotation of the Earth (C_e).

Taking D/Dt of this equation gives:

$$\frac{DC}{Dt}=\frac{DC_a}{Dt}-2\Omega\frac{DA_e}{Dt}$$

$$\frac{DC}{Dt}=-\oint\frac{dp}{\rho}-2\Omega\frac{DA_e}{Dt}$$

For a barotropic fluid this reduces to:

$$\frac{DC}{Dt} = -2\Omega \frac{DA_e}{Dt}$$

Integrating from an initial state (1) to a final state (2) gives:

$$C_2 - C_1 = -2\Omega\left(A_{e2} - A_{e1}\right)$$
$$= -2\Omega\left(A_2 \sin\phi_2 - A_1 \sin\phi_1\right)$$

What is the physical interpretation of this result?

Vorticity

Vorticity – a microscopic measure of rotation at any point in a fluid

Vorticity is defined as the curl of the velocity ($\nabla \times \vec{V}$)

Absolute vorticity: $\vec{\omega}_a = \nabla \times \vec{U}_a$

Relative vorticity: $\vec{\omega} = \nabla \times \vec{U}$

$$\vec{\omega} = \nabla \times \vec{U} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$
$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

It is typical to consider only the vertical component of the vorticity vector:

Vertical component of absolute vorticity: $\eta \equiv \vec{k} \cdot \left(\nabla \times \vec{U}_a\right)$

Vertical component of relative vorticity: $\zeta \equiv \vec{k} \cdot \left(\nabla \times \vec{U}\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Under what conditions will ζ be positive?

What is the relationship between absolute and relative voriticity?

Vertical component of planetary vorticity: $k \cdot \left(\nabla \times \vec{U}_e\right) = 2\Omega \sin\phi = f$

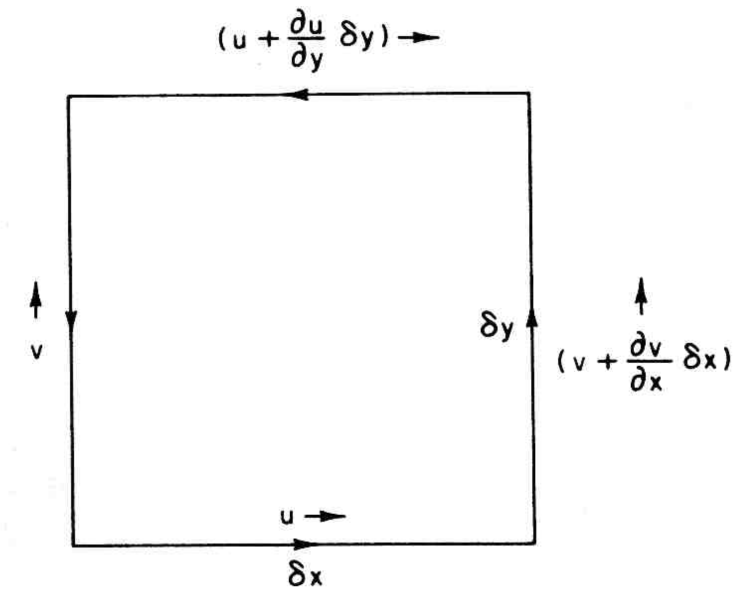
Then:

$$\eta = \zeta + f$$

The absolute vorticitiy is equal to the sum of the relative and planetary vorticity.

What is the relationship between vorticity and circulation?

Consider the circulation for flow around a rectangular area $\delta x \delta y$:



$$\delta C = u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\delta y$$

$$\delta C = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x \delta y$$

$$\delta C = \zeta \delta A$$

$$\zeta = \frac{\delta C}{\delta A}$$

We can also relate circulation and vorticity using Stokes' theorem and the definition of vorticity:

$$C = \oint \vec{U} \cdot d\vec{l}$$

$$C = \iint_A \left(\nabla \times \vec{U}\right) \cdot \vec{n} dA$$

$$C = \iint_A \zeta dA$$

$$C = \bar{\zeta} A$$

$$\bar{\zeta} = \frac{C}{A}$$

This indicates that the average vorticity is equal to the circulation divided by the area enclosed by the contour used in calculating the circulation.

For solid body rotation:

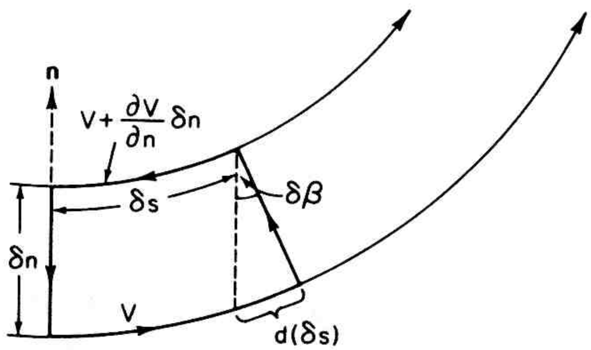
$$C = 2\Omega \pi R^2$$

and

$$\zeta = 2\Omega$$

In this case the vorticity (ζ) is twice the angular velocity (Ω)

Vorticity in Natural Coordinates



From this figure the circulation is:

$$\delta C = V\left[\delta s + d(\delta s)\right] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

$$\delta C = Vd(\delta s) - \frac{\partial V}{\partial n}\delta n \delta s$$

From the geometry shown in this figure:

$d(\delta s) = \delta\beta\delta n$, where $\delta\beta$ is the angular change in wind direction

This then gives:

$$\delta C = V\delta\beta\delta n - \frac{\partial V}{\partial n}\delta n\delta s$$

$$\delta C = \left(V\frac{\delta\beta}{\delta s} - \frac{\partial V}{\partial n}\right)\delta n\delta s$$

$$\frac{\delta C}{\delta n\delta s} = V\frac{\delta\beta}{\delta s} - \frac{\partial V}{\partial n}$$

In the limit as $\delta n, \delta s \rightarrow 0$:

$$\frac{\delta C}{\delta n\delta s} = \zeta = V\frac{\partial\beta}{\partial s} - \frac{\partial V}{\partial n}$$

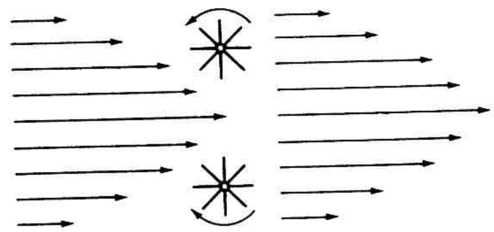
Noting that $\frac{\partial\beta}{\partial s} = \frac{1}{R_s}$ then

$$\zeta = \frac{V}{R_s} - \frac{\partial V}{\partial n}$$

What is the physical interpretation of this equation?

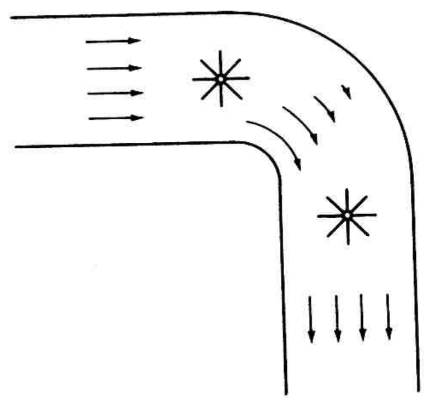
Can purely straigthline flow have non-zero vorticity?

Where would this occur in the real world?



(a)

Can curved flow have zero vorticity?



(b)

Potential Vorticity

Kelvin's circulation theorem states that circulation is constant in a barotropic fluid.

Can we expand this result to apply for less restrictive conditions?

Density can be expressed as a function of p and θ using the ideal gas law and the definition of θ.

$$\rho = \frac{p}{RT} \text{ and } \theta = T\left(\frac{p_s}{p}\right)^{R/c_p} \Rightarrow T = \theta\left(\frac{p}{p_s}\right)^{R/c_p}$$

Then:

$$\rho = \frac{p p_s^{R/c_p}}{R\theta p^{R/c_p}} = \frac{p^{\left(1-R/c_p\right)} p_s^{R/c_p}}{R\theta} = \frac{p^{c_v/c_p} p_s^{R/c_p}}{R\theta}$$

On an isentropic surface ρ is only a function of p and the solenoidal term is given by:

$$\oint \frac{dp}{\rho} \propto \oint dp^{\left(1-c_v/c_p\right)} = 0$$

Then for adiabatic motion a closed chain of fluid parcels on an isentropic surface satisfies Kelvin's circulation theorem:

$$\frac{DC_a}{Dt} = \frac{D\left(C + 2\Omega\delta A \sin\phi\right)}{Dt} = 0$$

For an approximately horizontal isentropic surface:

$$C \approx \zeta_\theta \delta A,$$

where ζ_θ is the relative vorticity evaluated on an isentropic surface.

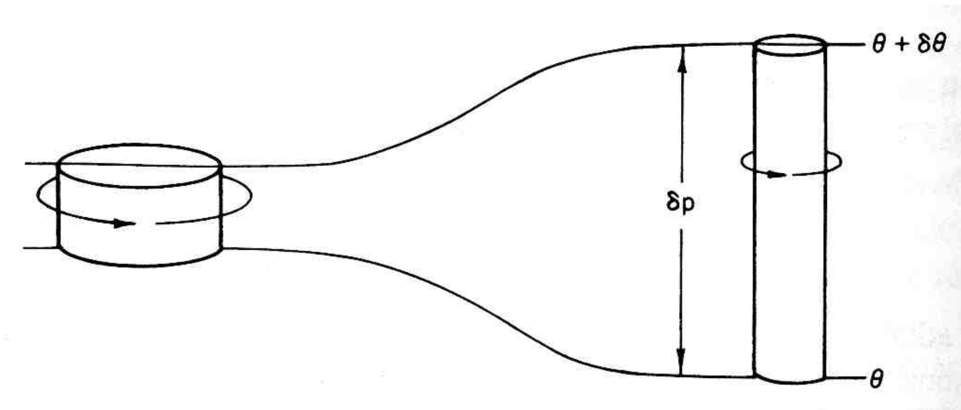
Kelvin's circulation theorem is then given by:

$$\frac{D\left(C + 2\Omega\delta A \sin\phi\right)}{Dt} = \frac{D\left[\delta A\left(\zeta_\theta + 2\Omega \sin\phi\right)\right]}{Dt} = \frac{D\left[\delta A\left(\zeta_\theta + f\right)\right]}{Dt} = 0$$

and

$\delta A\left(\zeta_\theta + f\right) = \mathrm{Const.}$ following the air parcel motion

Consider an air parcel of mass δM confined between two θ surfaces:



From the hydrostatic equation $\frac{\delta p}{\delta z} = -\rho g$ and $\delta M = \rho \delta V = \rho \delta z \delta A$:

$$\delta M = -\frac{\delta p}{g}\delta A$$

For δM conserved following the air parcel motion $\delta M = -\frac{\delta p}{g}\delta A = \text{Const.}$

Then:

$$\delta A = -\frac{\delta M g}{\delta p} = \left(-\frac{\delta \theta}{\delta p}\right)\left(\frac{\delta M g}{\delta \theta}\right) = (\text{Const.})g\left(-\frac{\delta \theta}{\delta p}\right)$$

where $\text{Const.} = \frac{\delta M}{\delta \theta}$

Using this expression to replace δA in $\delta A(\zeta_\theta + f) = \text{Const.}$ gives:

$$(\text{Const.})g\left(-\frac{\delta \theta}{\delta p}\right)(\zeta_\theta + f) = \text{Const.}$$

Ertel's Potential Vorticity (*P*) is defined as:

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\delta\theta}{\delta p}\right)$$

Ertel's potential vorticity has units of K kg^{-1} m^2 s^{-1} and is typically given as a potential vorticity unit (PVU = 10^{-6} K kg^{-1} m^2 s^{-1})

What is the sign of *P* in the Northern hemisphere?

For adiabatic, frictionless flow *P* is conserved following the motion.

The term potential vorticity is used for several similar quantities, but in all cases it refers to a quantity which is the ratio of the absolute vorticity to the vortex depth.

For Ertel's potential vorticity the vortex depth is given by $-\frac{\delta p}{\delta\theta}$

How will $\zeta_\theta + f$ change if the vortex depth increases (decreases)?

Potential vorticity for an incompressible fluid

For an incompressible fluid ρ is constant and the mass, *M*, is given by:

$M = \rho h \delta A$,

where *h* is the depth of the fluid being considered

Rearranging gives: $\delta A = \frac{M}{\rho h}$

And the potential vorticity is given by: $\frac{\zeta + f}{h}$

As above, the potential voriticity is constant for frictionless flow following the air parcel motion.

For a constant depth fluid:

$\zeta + f = \eta = \text{Const.}$ (absolute vorticity is conserved following the motion)

This puts a strong constraint on the types of motion that are possible.

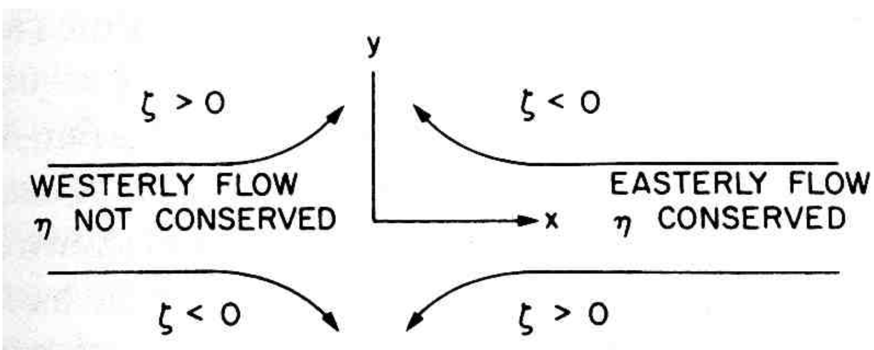
Consider a zonal flow that has $\zeta_0 = 0$ initially so $\eta_0 = f_0$

Since $\zeta + f = \eta = \text{Const.}$, then at some later time $\zeta + f = \eta_0 = f_0$

For a flow that turns to the north $f > f_0$ and $\zeta = f_0 - f < 0$

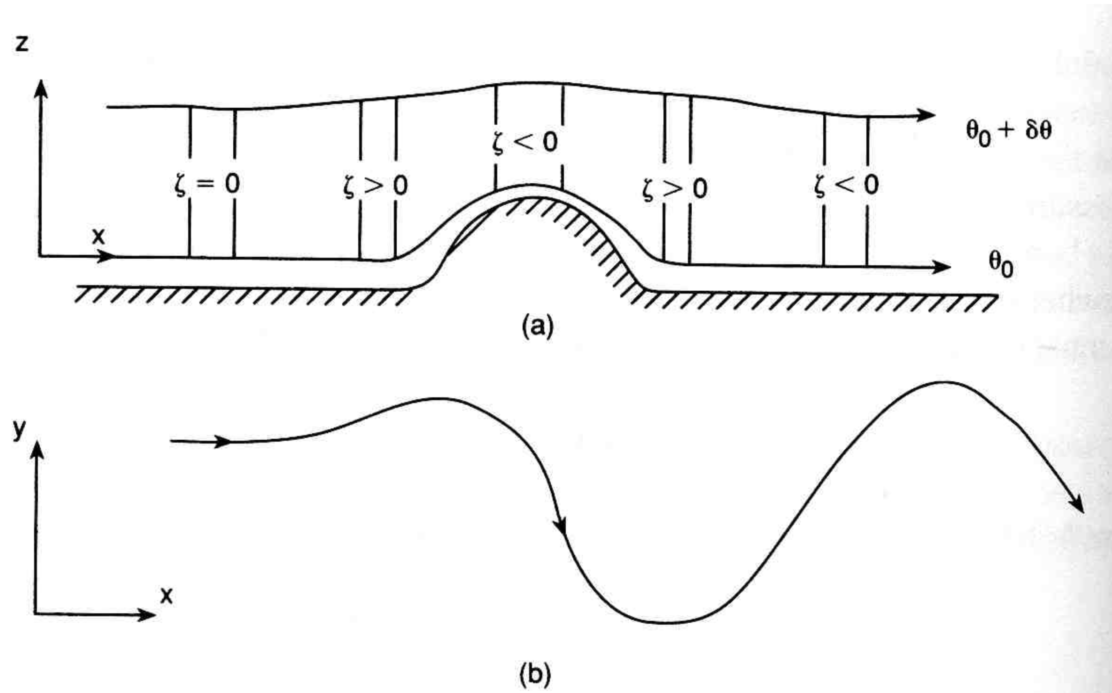
For a flow that turns to south $f < f_0$ and $\zeta = f_0 - f > 0$

Can these conditions be satisfied for both easterly and westerly flow?



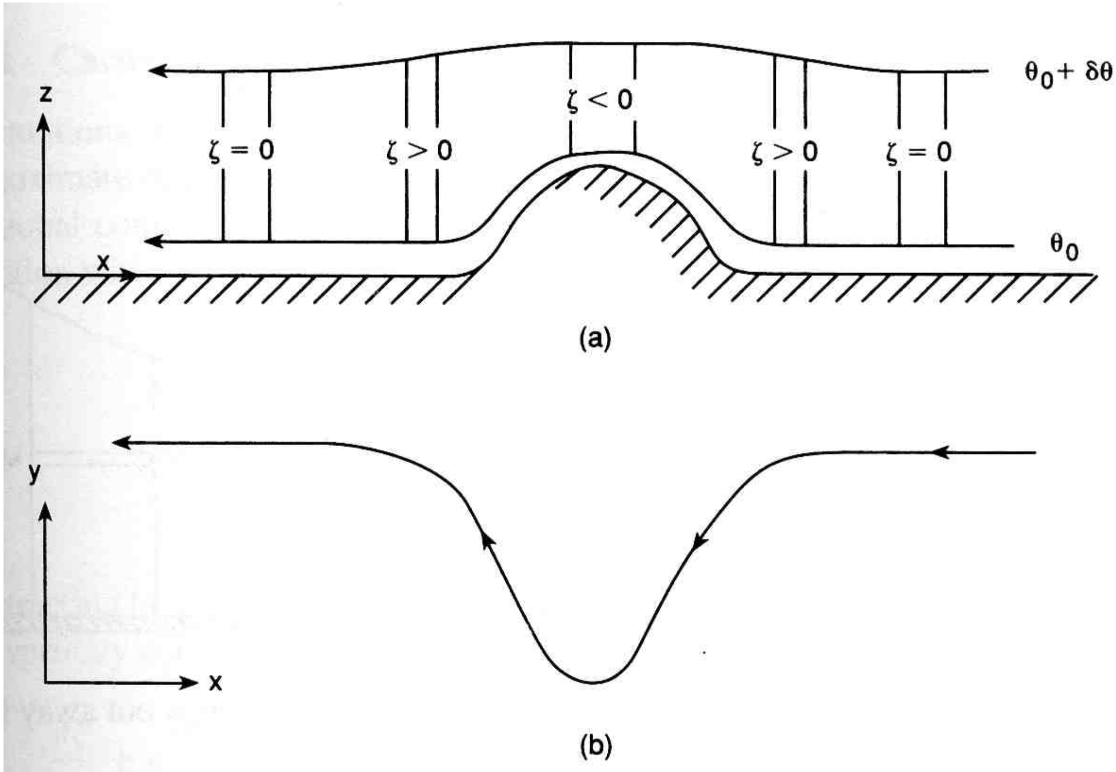
For a fluid in which the depth can vary potential vorticity, rather than absolute voriticity, is conserved.

Consider westerly flow over a mountain:



Air column depth	Increase $\lvert\delta p\rvert\uparrow$	Decrease $\lvert\delta p\rvert\downarrow$	Increase $\lvert\delta p\rvert\uparrow$	Decrease (return to original value)
Change in $\left\lvert\frac{\delta\theta}{\delta p}\right\rvert$	Decrease	Increase	Decrease	Increase (return to original value)
Change in $\lvert\zeta + f\rvert$	Increase	Decrease	Increase	Decrease
Sign of ζ	Positive	Negative	Positive	Negative
Resulting motion	northward	southward	lee side trough	southward
Change in $\lvert f\rvert$	Increase	Decrease	Increase	Decrease

For easterly flow over a mountain:



Air column depth	Decrease (Return to original value)	Increase $\lvert\delta p\rvert\downarrow$	Decrease $\lvert\delta p\rvert\uparrow$	Increase $\lvert\delta p\rvert\uparrow$
Change in $\left\lvert\frac{\delta\theta}{\delta p}\right\rvert$	Increase (Return to original value)	Decrease	Increase	Decrease
Change in $\lvert\zeta + f\rvert$	Decrease	Increase	Decrease	Increase
Sign of ζ	Zero	Positive	Negative	Positive
Resulting motion	westward	northward	southward	southward
Change in $\lvert f\rvert$	Return to original value	Increase	Decrease	Decrease

As was the case for flows in which absolute vorticity was conserved the flow responds differently for easterly and westerly flow.

Vorticity Equation

We are interested in being able to predict the time rate of change of vorticity without the constraints of adiabatic motion.

To do this we will derive the vorticity equation.

Since vorticity is defined as: $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

We will subtract $\frac{\partial}{\partial y}$ of the *u* momentum equation from $\frac{\partial}{\partial x}$ of the *v* momentum equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

This gives:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Noting that:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} = v\frac{\partial f}{\partial y}$$
$$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z}$$

the vorticity equation becomes:

$$\frac{D(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

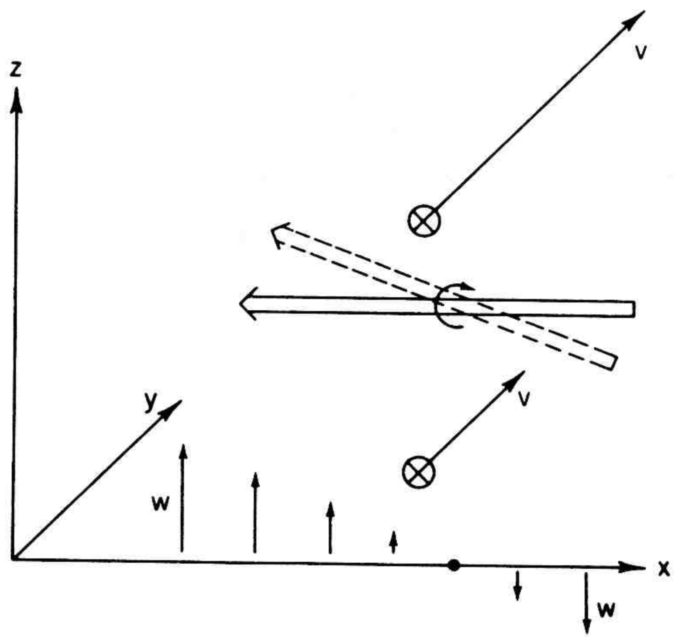
This equation indicates that changes in absolute vorticity following the motion is due to a:

Convergence term
Tilting term
Solenoidal term

What is the physical interpretation of each of these terms?

How will the vorticity change under the influence of convergence?

Physical interpretation of the tilting term:



What is the sign of $\frac{\partial w}{\partial x}$ and $\frac{\partial v}{\partial z}$ in this example?

Does this lead to an increase or decrease in absolute vorticity?

The solenoidal term is the microscopic equivalent of the solenoidal term from the circulation theorem.

From the circulation theorem the solenoidal term is given by:

$$-\oint \frac{dp}{\rho} = -\oint \alpha \nabla p \cdot d\vec{l}$$

Applying Stokes' theorem $\left[\oint \vec{F} \cdot d\vec{s} = \iint_A \left(\nabla \times \vec{F}\right) \cdot \vec{n} dA\right]$ to this gives:

$$-\oint \alpha \nabla p \cdot d\vec{l} = -\iint_A \left(\nabla \times \alpha \nabla p\right) \cdot \vec{k} dA$$

Using the vector identity $\nabla \times \alpha \nabla p = \nabla \alpha \times \nabla p$ gives:

$$-\oint \alpha \nabla p \cdot d\vec{l} = -\iint_A \left(\nabla \alpha \times \nabla p\right) \cdot \vec{k} dA$$

Expansion of the cross product gives:

$$-\left(\nabla \alpha \times \nabla p\right) \cdot \vec{k} = -\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}$$

Since:

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \rho^{-1}}{\partial x} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}$$

$$\frac{\partial \alpha}{\partial y} = \frac{\partial \rho^{-1}}{\partial y} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}$$

this gives:

$$-\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} = \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$

Then

$$-\iint_A (\nabla\alpha \times \nabla p)\cdot \vec{k}\,dA = \left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)A$$

Dividing this by the area gives the solenoidal term in the vorticity equation.

Vorticity equation in isobaric coordinates

$$\frac{\partial\zeta}{\partial t} = -\vec{V}\cdot\nabla(\zeta+f) - \omega\frac{\partial\zeta}{\partial p} - (\zeta+f)\nabla\cdot\vec{V} + \vec{k}\cdot\left(\frac{\partial\vec{V}}{\partial p}\times\nabla\omega\right)$$

What do each of the terms in this equation represent?

In isobaric coordinates there is no solenoidal term.

Scale Analysis of the Vorticity Equation

We can simplify the vorticity equation through scale analysis.

Consider the scales for mid-latitude synoptic weather systems:

Horizontal scale	U	$10\ \mathrm{m\ s^{-1}}$
Vertical scale	W	$10^{-2}\ \mathrm{m\ s^{-1}}$
Length scale	L	$10^{6}\ \mathrm{m}$
Depth scale	H	$10^{4}\ \mathrm{m}$
Horizontal pressure scale	δp	10 hPa
Horizontal pressure scale	$\delta p/\rho$	$10^{3}\ \mathrm{m^2\ s^{-2}}$
Mean density	ρ	$1\ \mathrm{kg\ m^{-3}}$
Fractional density fluctuation	$\delta\rho/\rho$	10^{-2}
Time scale	$T = L/U$	$10^{5}\ \mathrm{s}$
Coriolis parameter	F	$10^{-4}\ \mathrm{s^{-1}}$
“Beta” parameter	β	$10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$

$$\frac{\partial f}{\partial y} = \beta = \frac{\partial(2\Omega \sin\phi)}{\partial y} = \frac{\partial\left(2\Omega \sin\frac{y}{a}\right)}{\partial y}$$

$$\beta = \frac{2\Omega}{a}\cos\frac{y}{a} = \frac{2\Omega}{a}\cos\phi$$

What is the magnitude of β in the mid-latitudes?

ζ scales as:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \lesssim \frac{U}{L} \sim 10^{-5}\ \mathrm{s}^{-1},$$

where $\lesssim$ is used since $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ may partially cancel

What is the relative magnitude of ζ compared to f_0?

$$\frac{\zeta}{f_0} \lesssim \frac{U}{f_0 L} \equiv Ro \sim 10^{-1},$$

so ζ is often an order of magnitude smaller than f_0 (the planetary vorticity)

Using this we can approximate the convergence term as:

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

(i.e. we can neglect the relative vorticity contribution to the convergence term)

This approximation is not valid near the center of intense areas of low pressure where:

$$\left|\frac{\zeta}{f}\right| \sim 1$$

and then both relative and planetary vorticity must be considered in the convergence term.

What is an example of a type of flow where we cannot neglect ζ?

Scale analysis of the terms in the vorticity equation gives:

$$\frac{\partial \zeta}{\partial t}, u\frac{\partial \zeta}{\partial x}, v\frac{\partial \zeta}{\partial y} \sim \frac{U^2}{L^2} \sim 10^{-10} s^{-2}$$

$$w\frac{\partial \zeta}{\partial z} \sim \frac{WU}{LH} \sim 10^{-11} s^{-2}$$

$$v\frac{\partial f}{\partial y} \sim U\beta \sim 10^{-10} s^{-2}$$

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim \frac{f_0 U}{L} \sim 10^{-9} s^{-2}$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \lesssim \frac{W}{L}\frac{U}{H} \sim 10^{-11} s^{-2}$$

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \lesssim \frac{\delta\rho}{\rho}\frac{\delta p}{\rho}\frac{1}{L^2} \sim 10^{-11} s^{-2}$$

Note that $\lesssim$ is used for the last three terms in this list since portions of each of these terms may partially cancel.

What is the largest term in the vorticity equation?

In order for the vorticity equation to be satisfied:

$$f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\sim 10^{-10}\ \text{s}^{-2} < 10^{-9}\ \text{s}^{-2}$$

which implies that the flow must be quasi non-divergent:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\sim 10^{-6}\ \text{s}^{-1} < \frac{U}{L}\sim 10^{-5}\ \text{s}^{-1}$$

How does the magnitude of the divergence compare to the magnitude of the relative and planetary vorticity?

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ f_0\right| \lesssim \frac{10^{-6}\ \text{s}^{-1}}{10^{-4}\ \text{s}^{-1}}\sim 10^{-2}\sim Ro^2$$

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ \zeta\right| \lesssim \frac{10^{-6}\ \text{s}^{-1}}{10^{-5}\ \text{s}^{-1}}\sim 10^{-1}\sim Ro$$

The horizontal divergence is two orders of magnitude smaller than the *f* and one order of magnitude smaller than ζ

Retaining the largest terms (10^{-10} s^{-2}) in the vorticity equation gives:

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+v\frac{\partial f}{\partial y}=-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\frac{D_h(\zeta+f)}{Dt}=-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

where

$$\frac{D_h}{Dt}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

What is the physical interpretation of the scaled vorticity equation?

Near the center of intense low pressure systems, where ζ and f have similar magnitude the scaled vorticity equation becomes:

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Why can cyclonic systems become more intense than anticyclonic systems?

This is consistent with our analysis of the gradient wind equation where we found that the intensity of high pressure systems was limited while the intensity of low pressure systems was not.

Is the scaled vorticity equation appropriate in the vicinity of fronts?

Near fronts: $L \sim 10^5$ m and $W \sim 10^{-1}$ m s^{-1}

Using these scales indicates that the vertical advection, tilting, and solenoidal terms in the vorticity equation may all be as large as the divergence term, and thus must all be retained.

Vorticity in Barotropic Fluids

Barotropic (Rossby) Potential Vorticity Equation

Consider an atmosphere that is:

Barotropic: $\rho = \rho(p)$

Incompressible: $\frac{D\rho}{Dt} = 0$

With a depth $h(x,y,t) = z_2 - z_1$, where z_2 and z_1 are the heights of the upper and lower boundaries of the atmosphere

For an incompressible fluid the continuity equation is:

$$\nabla \cdot \vec{U} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Then the scaled vorticity equation becomes:

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = (\zeta + f)\frac{\partial w}{\partial z}$$

Approximating the vorticity (ζ) by the geostrophic vorticity (ζ_g), where

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

gives:

$$\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{\partial w}{\partial z}$$

In a barotropic fluid $\vec{V}_g$ does not vary with height and the equation above can be integrated from height z_1 to z_2 to give:

$$\int_{z_1}^{z_2} \frac{D_h(\zeta_g + f)}{Dt}\partial z = \int_{z_1}^{z_2} (\zeta_g + f)\partial w$$

$$\frac{D_h(\zeta_g + f)}{Dt}(z_2 - z_1) = (\zeta_g + f)[w(z_2) - w(z_1)]$$

$$h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)[w(z_2) - w(z_1)]$$

Noting that $w \equiv \frac{D_h z}{Dt}$ gives:

$w(z_1) = \frac{D_h z_1}{Dt}$ and $w(z_2) = \frac{D_h z_2}{Dt}$

so

$$w(z_2) - w(z_1) = \frac{D_h z_2}{Dt} - \frac{D_h z_1}{Dt} = \frac{D_h (z_2 - z_1)}{Dt} = \frac{D_h h}{Dt}$$

Then:

$$h \frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{D_h h}{Dt}$$

$$\frac{1}{(\zeta_g + f)} \frac{D_h(\zeta_g + f)}{Dt} = \frac{1}{h}\frac{D_h h}{Dt}$$

$$\frac{D_h \ln(\zeta_g + f)}{Dt} = \frac{D_h \ln h}{Dt}$$

$$\frac{D_h\left[(\zeta_g + f)/h\right]}{Dt} = 0$$

Rossby potential vorticity: $(\zeta_g + f)/h$

The Rossby potential vorticity is conserved following the motion.

Barotropic Vorticity Equation

For a barotropic, incompressible fluid with purely horizontal flow ($w = 0$) the vorticity equation becomes:

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\frac{\partial w}{\partial z} = 0$$

and the absolute vorticity ($\zeta + f$) is conserved following the motion.

A more general result is that for a flow that has non-divergent horizontal winds $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\right)$:

$$\frac{D_h(\zeta+f)}{Dt}=0$$

For a non-divergent flow the wind can be represented by a streamfunction $[\psi(x,y)]$ such that:

$$\vec{V}_\psi \equiv \vec{k}\times\nabla\psi=-\frac{\partial\psi}{\partial y}\vec{i}+\frac{\partial\psi}{\partial x}\vec{j}=u\vec{i}+v\vec{j}$$

Example: Show that with this definition of the streamfunction the horizontal wind is non-divergent.

The relative vorticity is then given by:

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\equiv\nabla^2\psi$$

The vorticity equation can then be written as:

$$\frac{D_h(\zeta+f)}{Dt}=0$$

$$\frac{D_h\left(\nabla^2\psi+f\right)}{Dt}=\frac{\partial\nabla^2\psi}{\partial t}+\vec{V}_\psi\cdot\nabla\left(\nabla^2\psi+f\right)=0,$$

$$\frac{\partial\nabla^2\psi}{\partial t}=-\vec{V}_\psi\cdot\nabla\left(\nabla^2\psi+f\right)$$

What is the physical interpretation of this equation?

This equation can be solved numerically to predict the time evolution of ψ.

V and ζ can then be calculated from ψ.

The flow in the mid-troposphere (~500 mb) is often nearly non-divergent and this equation can be used to produce a reasonable prediction of the time evolution of the flow at this level.

Baroclinic (Ertel) Potential Vorticity Equation

Ertel potential vorticity is given by:

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

and is conserved following the motion for adiabatic, frictionless flow.

We want to derive an equation for the time evolution of *P* in the presence of diabatic effects and friction.

This equation can be derived by rewriting the horizontal momentum equation in isentropic coordinates.

We can use θ as a vertical coordinate in a stably stratified atmosphere since $\partial\theta/\partial z > 0$ and θ is a monotonic function of height.

The vertical velocity in isentropic coordinates is defined as:

$$\dot{\theta} \equiv \frac{D\theta}{Dt}$$

When will $\dot{\theta} = 0$?

For an air parcel with mass δM:

$$\delta M = \rho\delta A\delta z = \delta A\left(-\frac{\delta p}{g}\right) = \frac{\delta A}{g}\left(-\frac{\partial p}{\partial\theta}\right)\delta\theta = \sigma\delta A\delta\theta,$$

where $\sigma = -\frac{1}{g}\frac{\partial p}{\partial\theta}$ is the density in (x,y,θ) space

Then the governing equations in isentropic coordinates are given by:

Horizontal momentum equation in isentropic coordinates:

$$\frac{\partial \vec{V}}{\partial t} + \nabla_\theta\left(\frac{\vec{V}\cdot\vec{V}}{2} + \Psi\right) + (\zeta_\theta + f)\vec{k}\times\vec{V} = -\dot{\theta}\frac{\partial \vec{V}}{\partial \theta} + \vec{F}_r,$$

where $\Psi = c_p T + \Phi$ is the Montgomery streamfunction

Continuity equation in isentropic coordinates:

$$\frac{\partial \sigma}{\partial t} + \nabla_\theta \cdot \left(\sigma \vec{V}\right) = -\frac{\partial\left(\sigma\dot{\theta}\right)}{\partial \theta}$$

Hydrostatic equation in isentropic coordinates:

$$\frac{\partial \Psi}{\partial \theta} = \Pi(p) \equiv c_p\left(\frac{p}{p_s}\right)^{R/c_p} = c_p\frac{T}{\theta},$$

where Π is the Exner function

Isentropic vorticity equation:

$$\frac{\tilde{D}(\zeta_\theta + f)}{Dt} + (\zeta_\theta + f)\nabla_\theta \cdot \vec{V} = \vec{k}\cdot\nabla_\theta \times\left(\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta}\right),$$

where $\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla_\theta$ is the total derivative following the horizontal motion on an isentropic surface

The continuity equation can be rewritten as:

$$\frac{\tilde{D}\sigma^{-1}}{Dt} - \sigma^{-1}\nabla_\theta\cdot\vec{V} = \sigma^{-2}\frac{\partial\left(\sigma\dot{\theta}\right)}{\partial \theta}$$

Ertel potential vorticity equation

$$\frac{\tilde{D}P}{Dt} = \frac{\partial P}{\partial t} + \vec{V} \cdot \nabla_\theta P = \frac{P}{\sigma}\frac{\partial\left(\sigma\dot{\theta}\right)}{\partial\theta} + \sigma^{-1}\vec{k} \cdot \nabla_\theta \times \left(\vec{F}_r - \dot{\theta}\frac{\partial\vec{V}}{\partial\theta}\right),$$

where Ertel's potential vorticity is given by $P = \left(\zeta_\theta + f\right)/\sigma$

What is the physical interpretation of this equation?

For adiabatic, frictionless flow $\frac{\tilde{D}P}{Dt} = 0$ and *P* is conserved following the motion.