Elementary Application of the Basic Equations

For many applications it is preferable to work with the equations that govern atmospheric motions using an isobaric coordinate system.

Basic Equations in Isobaric Coordinates

Horizontal Momentum Equation

We will neglect the curvature terms, friction force, and Coriolis force due to vertical motion.

In height coordinates the horizontal momentum equation is then given by:

$$\frac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} = -\frac{1}{\rho}\nabla p,$$

where $\vec{V} = u\vec{i} + v\vec{j}$ is the horizontal wind vector.

Replace the pressure gradient force term with:

$$-\frac{1}{\rho}\nabla p = -\nabla_p \Phi = -\frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial y}$$

In an isobaric coordinate system:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p},$$

where $\omega \equiv \dfrac{Dp}{Dt}$

How does this differ from the expression for a total derivative in a height based coordinate system?

What is the physical interpretation of ω?

The horizontal momentum equation in an isobaric coordinate system is:

$$\frac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} = -\nabla_p \Phi$$

The geostrophic relationship in isobaric coordinates is:

$$f\vec{k} \times \vec{V}_g = -\nabla_p \Phi$$

$$\vec{V}_g = \frac{1}{f}\vec{k} \times \nabla_p \Phi$$

$$\vec{V}_g = u_g\vec{i} + v_g\vec{j} = -\frac{1}{f}\frac{\partial \Phi}{\partial y}\vec{i} + \frac{1}{f}\frac{\partial \Phi}{\partial x}\vec{j}$$

One advantage of the isobaric coordinate system is that the geostrophic wind does not depend on density, and thus a given geopotential gradient results in the same geostrophic wind at all heights in the atmosphere.

For constant *f* the divergence of the geostrophic wind is zero.

Example: Show that $\nabla_p \cdot \vec{V}_g = 0$.

Continuity Equation

We will derive the continuity equation (conservation of mass) using a Lagrangian control volume ($\delta V = \delta x \delta y \delta z$) of constant mass ($\delta M = \rho \delta x \delta y \delta z$).

Use the hydrostatic equation to replace δz:

$$\frac{\delta p}{\delta z} = -\rho g$$

$$\delta z = -\frac{\delta p}{\rho g}$$

Then: $\delta M = -\delta x \delta y \frac{\delta p}{g}$

$$\frac{1}{\delta M}\frac{D(\delta M)}{Dt} = \frac{g}{\delta x \delta y \delta p}\frac{D(\delta x \delta y \delta p / g)}{Dt} = 0$$

$$= \frac{1}{\delta x}\frac{D(\delta x)}{Dt} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta p}\frac{D(\delta p)}{Dt} = 0$$

$$= \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

In the limit as $\delta x, \delta y, \delta p \rightarrow 0$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

How does this form of the continuity equation compare to the continuity equation used in a height coordinate system?

Thermodynamic Energy Equation

Start with: $c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J$ (First Law of Thermodynamics)

Noting that $\frac{Dp}{Dt} = \omega$ and $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}$ gives:

$$c_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right] - \alpha\omega = J$$

Using $\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \frac{T}{\theta}\frac{\partial \theta}{\partial p}$ and defining a static stability parameter (S_p) as:

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}$$

gives:

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = \frac{J}{c_p}$$

In the static stability parameter the term $\partial\theta/\partial p$ in isobaric coordinates is equivalent to $\partial\theta/\partial z$ in height coordinates.

The static stability parameter can be rewritten as:

$$S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{(\Gamma_d - \Gamma)}{\rho g}$$

Under what conditions is S_p positive?

What does this imply about the sign of $\partial\theta/\partial p$ and how θ varies in the vertical?

From this expression for S_p we note that S_p depends on $1/\rho$ and thus will increase rapidly with height in the atmosphere.

Balanced Flows

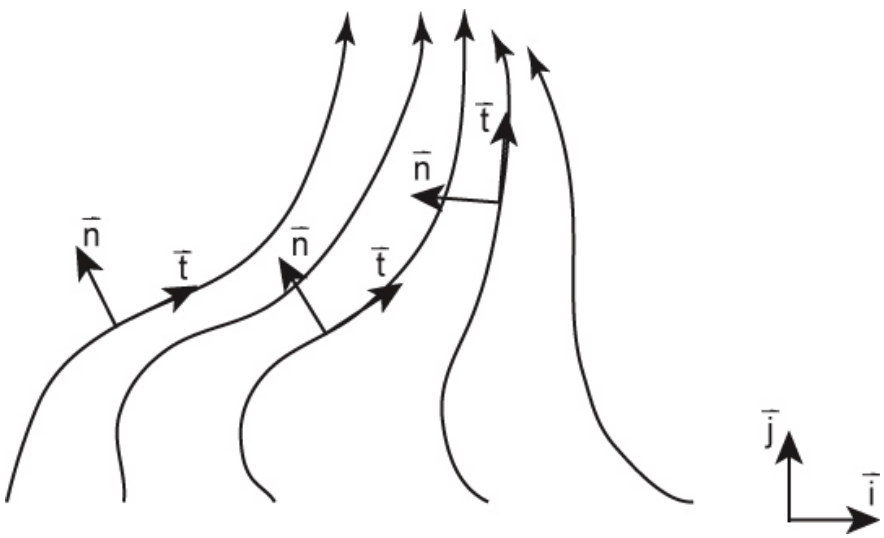
Many atmospheric flows can be understood by assuming a simple force balance.

In the next section we will consider steady state flow with no vertical component.

Natural Coordinates

Natural coordinate system – a coordinate system in which one axis is always tangent to the horizontal wind ($\vec{t}$) and a second axis is always normal to and to the left of the wind ($\vec{n}$).

In this coordinate system the vertical unit vector is $\vec{k}$.



Coordinate locations in natural coordinates: (s,n,z)

In natural coordinates the horizontal wind vector is given by:

$$\vec{V} = V\vec{t},$$

V is the horizontal wind speed and is defined to be non-negative.

$$V \equiv \frac{Ds}{Dt},$$

where *s* is the distance along the curve followed by the air parcel [*s*(*x*,*y*,*t*)]

The acceleration of the wind in natural coordinates is given by:

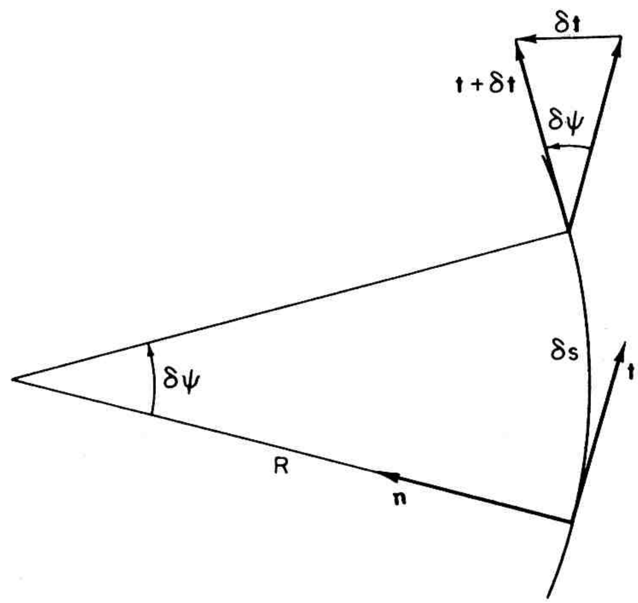
$$\frac{D\vec{V}}{Dt} = \frac{D(V\vec{t})}{Dt} = \vec{t}\frac{DV}{Dt} + V\frac{D\vec{t}}{Dt}$$

We will use this expression when writing the full equations of motion, but first we must derive an expression for $\frac{D\vec{t}}{Dt}$.

Consider the geometry for an air parcel moving a distance of δs along a curve.

From this geometry we can derive an expression for:

$$\frac{D\vec{t}}{Dt}$$



$$\delta\psi = \frac{\delta s}{|R|} = \frac{|\delta\vec{t}|}{|\vec{t}|} = |\delta\vec{t}|$$

$$|\delta\vec{t}| = \frac{\delta s}{|R|}$$

R is the radius of curvature following the air parcel motion.

R is taken to be positive when the center of curvature is to the left of the flow (in the positive $\vec{n}$ direction).

Dividing this expression by δs and taking the limit as $\delta s \rightarrow 0$ gives:

$$\frac{\delta\vec{t}}{\delta s} = \frac{1}{|R|},$$

$$\frac{D\vec{t}}{Ds} = \frac{\vec{n}}{R}$$

where we have noted that the direction of $\frac{D\vec{t}}{Ds}$ is parallel to $\vec{n}$.

This can now be used to express $\frac{D\vec{t}}{Dt}$ as:

$$\frac{D\vec{t}}{Dt} = \frac{D\vec{t}}{Ds}\frac{Ds}{Dt} = V\frac{\vec{n}}{R}$$

Then the acceleration of the wind in natural coordinates is given by:

$$\frac{D\vec{V}}{Dt} = \vec{t}\frac{DV}{Dt} + \frac{V^2}{R}\vec{n}$$

What is the physical interpretation of the terms in this equation?

Pressure Gradient Force in Natural Coordinates

$$-\nabla_p \Phi = -\frac{\partial \Phi}{\partial s}\vec{t} - \frac{\partial \Phi}{\partial n}\vec{n}$$

Coriolis Force in Natural Coordinates

$$-f\vec{k} \times \vec{V} = -fV\vec{n}$$

Component form of the horizontal momentum equation in natural coordinates:

s component: $\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$

n component: $\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$

For flow parallel to height contours:

$\frac{\partial \Phi}{\partial s} = 0$, so $\frac{DV}{Dt} = 0$ and the wind speed is constant.

Inertial Flow

What force balance will occur if there is no horizontal pressure gradient?

$$\frac{V^2}{R} + fV = 0$$

What is the path followed by an air parcel for this force balance?

Solving for the radius of curvature for this flow gives:

$$R = -\frac{V}{f}$$

What is the sign of *R* in the Northern and Southern hemispheres?
What does this imply about the direction of the flow?
The time required for an air parcel to complete one revolution around a path of radius *R* is given by:

$$\text{Period} = \left|\frac{2\pi R}{V}\right| = \left|\frac{-2\pi V}{fV}\right| = \left|\frac{2\pi}{f}\right| = \frac{2\pi}{|2\Omega \sin\phi|} = \frac{\frac{1}{2}\,\text{day}}{|\sin\phi|}$$

Where is this type of motion most likely to be observed?

Cyclostrophic Flow

The Rossby number is the ratio of the acceleration of the wind to the Coriolis force.

In natural coordinates the Rossby number is given by:

$$Ro = \frac{V^2/R}{fV} = \frac{V}{fR}$$

When will *Ro* be large?

What does this imply about the importance of the Coriolis force in the horizontal momentum equation?

For this case the horizontal equation of motion becomes:

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}$$

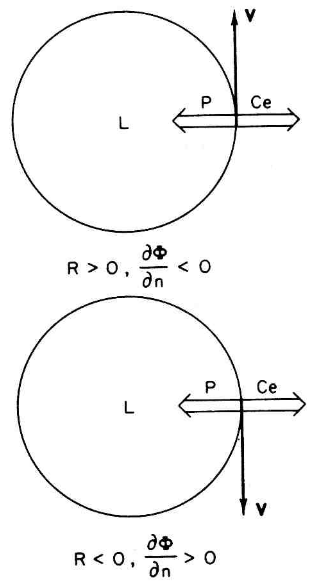
What force balance does this represent?

The wind that satisfies this equation is known as the cyclostrophic wind:

$$V = \left(-R\frac{\partial \Phi}{\partial n}\right)^{0.5}$$

A real solution of this equation requires $-R\frac{\partial \Phi}{\partial n} \geq 0$.

Can cyclostrophic flow occur around both low and high pressure centers?



Example: A tornado is observed to have a radius of 600 m and a wind speed of 130 m s^{-1}.

What is the Rossby number for this flow?

Is a cyclostrophic balance appropriate for these conditions?

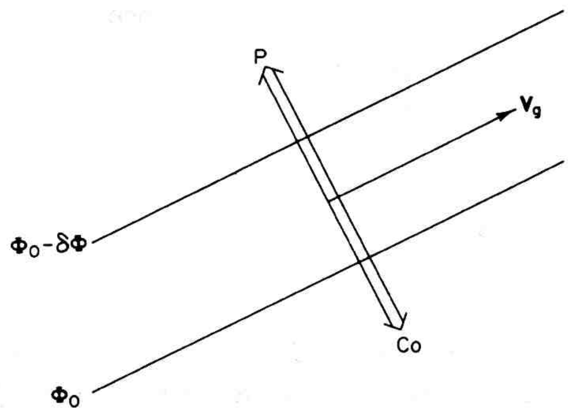
What is the pressure at the center of the tornado, assuming that the pressure at a distance of 600 m from the center is 1000 mb?

Geostrophic Flow in Natural Coordinates

For geostrophic flow the Coriolis force and pressure gradient force balance:

$$fV_g = -\frac{\partial \Phi}{\partial n}$$

$$V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$$



For this balance $\frac{V^2}{R} = 0$.

Under what conditions will $\frac{V^2}{R} \rightarrow 0$?

Example: Calculate the geostrophic wind from a constant pressure map.

Gradient Wind Approximation

What type of flow will occur when *Ro* is small?

What form of the horizontal momentum equation needs to be considered when *Ro*~1?

Gradient wind – a horizontal, frictionless flow that is parallel to the height contours on a constant pressure map

For the gradient wind the horizontal momentum equation is:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

This quadratic equation can be solved for the gradient wind (*V*):

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - R\frac{\partial \Phi}{\partial n} \right)^{0.5}$$

The gradient wind equation can also be written using V_g to replace $-\frac{1}{f}\frac{\partial \Phi}{\partial n}$ to give:

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} + fV_g R \right)^{0.5}$$

Multiple solutions exist for these equations, but not all of the solutions are physically reasonable.

What are the requirements for a physically reasonable solution to the gradient wind equation?

	Northern Hemisphere			
	CCW flow around L	**CW flow around H**	**CW flow around L**	**CCW flow around H**
f	+	+	+	+
R	+	-	-	+
$\frac{\partial \Phi}{\partial n}$	-	-	+	+
$\left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{0.5}$	always $> \frac{fR}{2}$	$< \frac{fR}{2}$ or imaginary for $\frac{f^2R^2}{4} < R\frac{\partial \Phi}{\partial n}$	always $> \frac{fR}{2}$	$< \frac{fR}{2}$ or imaginary for $\frac{f^2R^2}{4} < R\frac{\partial \Phi}{\partial n}$
$-\frac{fR}{2}$	-	+	+	-
V positive for:	+ root only	either root but $\frac{f^2R^2}{4} > R\frac{\partial \Phi}{\partial n}$	+ root only	never
Type of flow: Positive Root	Regular low	Anomalous High	Anomalous Low	No physical solution
Type of flow: Negative Root	No physical solution	Regular High	No physical solution	No physical solution

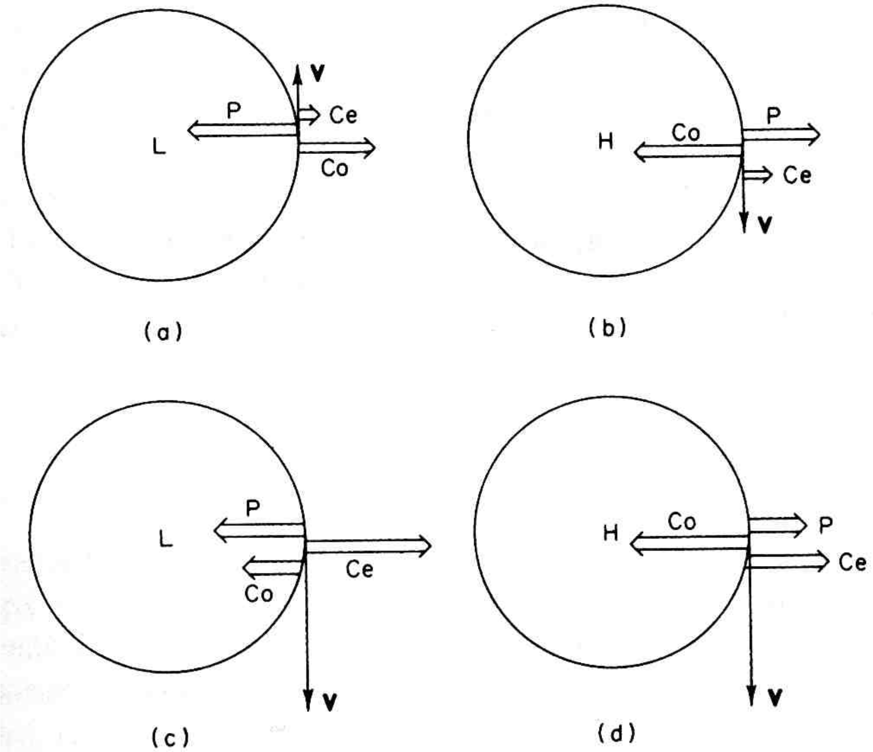
Baric flow – pressure gradient and Coriolis force act in opposite directions
Antibaric flow – pressure gradient and Coriolis force act in the same direction

Which gradient wind solutions are baric / antibaric?

Cyclonic flow – flow in which $Rf > 0$
Anticyclonic flow – flow in which $Rf < 0$

Which gradient wind solutions are cyclonic / anticyclonic?

Regular Low | Regular High



Anomalous Low | Anomalous High

For the anomalous low and anomalous high cases the gradient wind solution does not reduce to the geostrophic solution for $R \to \infty$.

The gradient wind equations for the regular low and regular high cases in the Northern Hemisphere are:

$$\text{Cyclonic (CCW) flow around L} \quad : \quad R > 0; \quad V = -\frac{fR}{2} + \left(\frac{f^2 R^2}{4} - R\frac{\partial \Phi}{\partial n} \right)^{0.5}$$

$$\text{Anticyclonic (CW) flow around H} \quad : \quad R < 0; \quad V = -\frac{fR}{2} - \left(\frac{f^2 R^2}{4} - R\frac{\partial \Phi}{\partial n} \right)^{0.5}$$

Example: Calculate the gradient wind from an upper air weather map

For both physically reasonable solutions for flow around a high pressure center the gradient wind solution requires that:

$$\frac{f^2R^2}{4} > R\frac{\partial\Phi}{\partial n}$$

This requires that as R decreases $\frac{\partial\Phi}{\partial n}$ (the pressure gradient) must also decrease.

What is the physical reason for this limit?

What does this imply about the weather that would be experienced near the center of a high pressure center?

Of the balanced flow cases considered above the gradient wind solution will provide the best estimate of the actual wind since it makes the fewest assumptions.

Using the gradient wind equation with the pressure gradient force expressed in terms of V_g allows us to derive an expression for the ratio of the geostrophic wind to the gradient wind:

$$\frac{V_g}{V} = 1 + \frac{V}{fR}$$

Under what conditions will $V_g > V$ ($V_g < V$)?

In this case the geostrophic wind overestimates (underestimates) the actual wind speed.

What is the physical explanation for this?

The ageostrophic wind is defined as:

$$\vec{V}_a = \vec{V} - \vec{V}_g$$

and is the part of the wind that is not in geostrophic balance.

Recall that the geostrophic wind is non-divergent, so divergence can only occur through the ageostrophic portion of the wind.

What does the difference between V_g and V imply about the ageostrophic wind in troughs and ridges?

What does this imply about the location of areas where converegence or divergence are occurring?

Example: Convergence and divergence on a constant pressure map.

Trajectories and Streamlines

Trajectory – the path followed by an air parcel over time

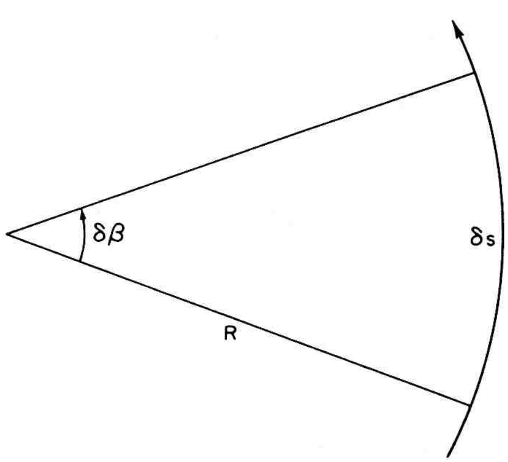
The radius of curvature, R, used in the previous section is the radius of curvature of the air parcel trajectory.

Streamline – a line that is everywhere parallel to the flow at a given time

For the balanced flow cases considered above the height contours are streamlines, since the flow is parallel to the height contours.

Often we will estimate the radius of curvature from height contours on a constant pressure map.

How does the radius of curvature estimated in this way differ from the radius of curvature of the air parcel trajectory?



Using this geometry:

$$\delta s = R\delta\beta$$

$$\frac{\delta\beta}{\delta s} = \frac{1}{R},$$

where β = angular wind direction,
s = distance along path, and
R = radius of curvature

Following the motion of an air parcel this gives:

$\frac{D\beta}{Ds} = \frac{1}{R_T}$, where R_T is the radius of curvature of a trajectory.

At a fixed time this geometry implies:

$\frac{\partial\beta}{\partial s} = \frac{1}{R_s}$, where R_s is the radius of curvature of a streamline

Noting that:

$$\frac{D\beta}{Dt} = \frac{D\beta}{Ds}\frac{Ds}{Dt} = \frac{V}{R_T}$$

and using the definition of the total derivative (*D/Dt*) gives:

$$\frac{D\beta}{Dt} = \frac{\partial\beta}{\partial t} + V \cdot \frac{\partial\beta}{\partial s}$$

$$\frac{V}{R_T} = \frac{\partial\beta}{\partial t} + \frac{V}{R_s} \quad ,$$

$$\frac{\partial\beta}{\partial t} = V\left(\frac{1}{R_T} - \frac{1}{R_s}\right)$$

where $\frac{\partial\beta}{\partial t}$ is the turning of the wind over time at a fixed location.

For steady state conditions $\frac{\partial\beta}{\partial t} = 0$ and $R_T = R_s$.

For non-steady state cases $R_T \neq R_s$.

Consider the case of an eastward moving trough:

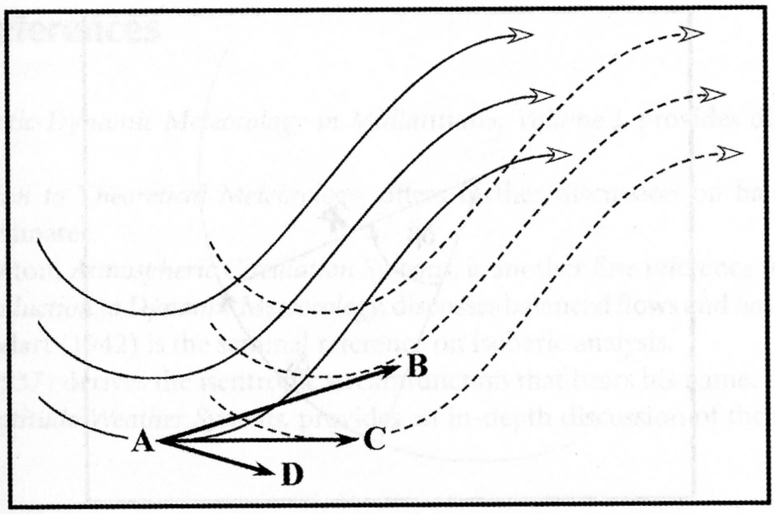
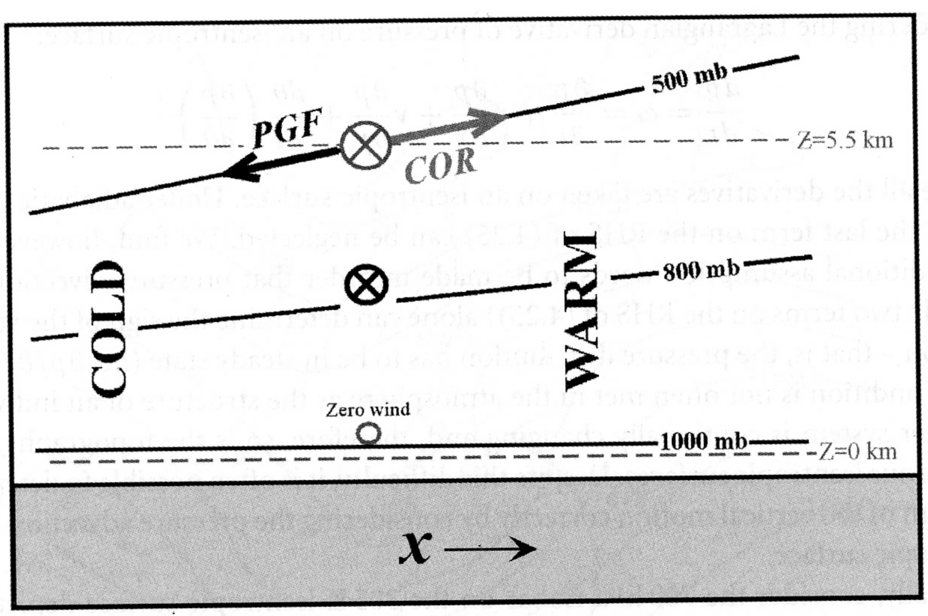


Figure 4.22 Streamlines and trajectories in an eastward-moving upper trough. Thin solid lines represent streamlines of the flow at some initial time while the dashed lines represent streamlines at some later time. The bold arrows (AD, AC, and AB) represent the trajectories of air parcels moving slower than the wave, at the same speed as the wave, and faster than the wave, respectively

Thermal Wind

How will horizontal temperature gradients alter the height, and thus slope, of upper level constant pressure surfaces?



Using the hypsometric equation we can relate the thickness (δz) between two pressure surfaces (e.g. 1000 and 500 mb) to the layer average temperature:

$$\delta z = -\frac{RT}{g}\delta \ln p$$

Assuming a flat lower pressure surface we will find that the upper pressure surface slopes down towards the cold air.

This slope results in a pressure gradient force ($-\nabla_p \Phi$).

This pressure gradient force can be used to calculate the geostrophic wind:

$$\vec{V}_g = \frac{1}{f}\left(\vec{k} \times \nabla_p \Phi\right)$$

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}, v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

What does the change in slope with height imply about changes in the geostrophic wind with height?

For the case illustrated above:

u_g = 0 at all levels and v_g = 0 at 1000 mb and is positive at upper levels.

We want to derive a relationship between the change in the geostrophic wind with height and the horizontal temperature gradient.

This relationship is known as the thermal wind relationship.

Take $\frac{\partial}{\partial p}$ of the equation for the geostrophic wind to get an equation that expresses the change in geostrophic wind with pressure (i.e. the change in geostrophic wind in the vertical direction).

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f}\frac{\partial^2 \Phi}{\partial y \partial p} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right) \qquad \frac{\partial v_g}{\partial p} = \frac{1}{f}\frac{\partial^2 \Phi}{\partial x \partial p} = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right)$$

From the hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\rho} = -g\,\partial z = -\partial \Phi$$

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p}$$

Then:

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f}\frac{\partial}{\partial y}\left(-\frac{RT}{p}\right)_p \qquad \frac{\partial v_g}{\partial p} = \frac{1}{f}\frac{\partial}{\partial x}\left(-\frac{RT}{p}\right)_p$$

$$p\frac{\partial u_g}{\partial p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \qquad p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p$$

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \qquad \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p$$

In vector form this gives the thermal wind equation:

$$\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R}{f}\left(\vec{k} \times \nabla_p T\right)$$

Thermal wind $\left(\vec{V}_T\right)$ is the vector difference between the geostrophic wind at two pressure levels

$$\vec{V}_T = \vec{V}_g(p_1) - \vec{V}_g(p_0)$$

Integrating the thermal wind equation from pressure level p_0 to pressure level p_1 gives:

$$\int_{p_0}^{p_1} \partial \vec{V}_g = \int_{p_0}^{p_1} -\frac{R}{f}\left(\vec{k} \times \nabla_p T\right) \partial \ln p$$

$$\vec{V}_g(p_1) - \vec{V}_g(p_0) \equiv \vec{V}_T = -\frac{R}{f}\vec{k} \times \nabla_p \langle T \rangle \ln\left(\frac{p_1}{p_0}\right)$$

or in component form:

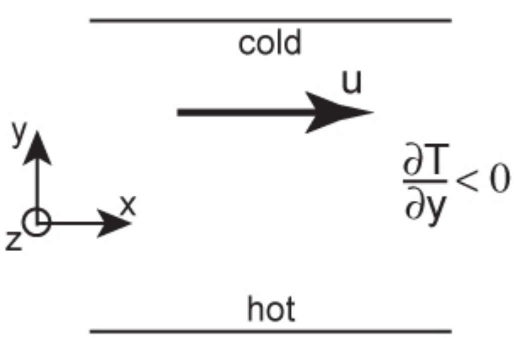
$$u_T \equiv u_g(p_1) - u_g(p_0) = -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right)$$

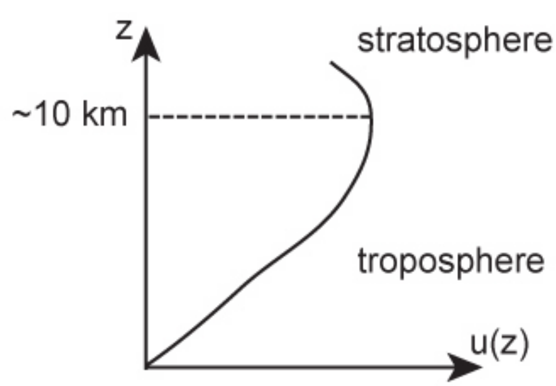
$$v_T \equiv v_g(p_1) - v_g(p_0) = \frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right),$$

where $\langle T \rangle$ is the layer average temperature

What is the physical interpretation of these equations?

How can we use these equations to explain the increase of westerly winds with height in the mid-latitude troposphere?





What is the relationship between the thermal wind and thickness lines on weather maps?

Integrate the hydrostatic equation $\left(\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}\right)$ from p_0 to p_1:

$$\int_{p_0}^{p_1} \partial \Phi = \int_{p_0}^{p_1} -\frac{RT}{p} \partial p$$

$$\Phi(p_1) - \Phi(p_0) = \Phi_1 - \Phi_0 = -R\langle T \rangle \ln\left(\frac{p_1}{p_0}\right) = R\langle T \rangle \ln\left(\frac{p_0}{p_1}\right),$$

$$\Phi_1 - \Phi_0 = g(Z_1 - Z_0) = gZ_T = R\langle T \rangle \ln\left(\frac{p_0}{p_1}\right)$$

where Z_T is the thickness between pressure levels p_1 and p_0.

From this we see that thickness (Z_T) is proportional to the layer mean temperature.

Substituting this into the thermal wind equation gives:

$$u_T = -\frac{1}{f}\frac{\partial(\Phi_1 - \Phi_0)}{\partial y} \qquad v_T = \frac{1}{f}\frac{\partial(\Phi_1 - \Phi_0)}{\partial x}$$

$$u_T = -\frac{g}{f}\frac{\partial Z_T}{\partial y} \qquad v_T = \frac{g}{f}\frac{\partial Z_T}{\partial x}$$

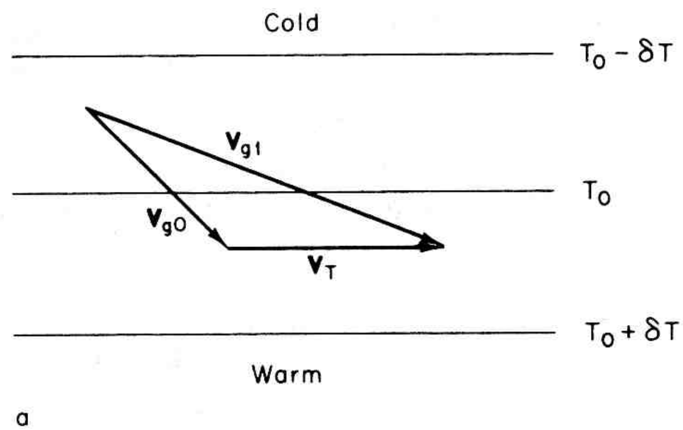
or in vector form:

$$\vec{V}_T = \frac{R}{f}\vec{k} \times \nabla\langle T \rangle \ln\left(\frac{p_0}{p_1}\right) = \frac{1}{f}\vec{k} \times \nabla(\Phi_1 - \Phi_0) = \frac{g}{f}\vec{k} \times \nabla Z_T$$

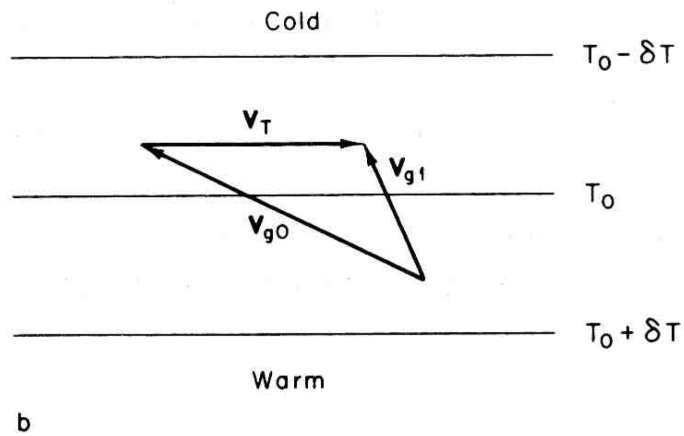
This equation indicates that the thermal wind is parallel to thickness lines (or isotherms of layer mean temperature) on a weather map.

What is the direction of $\vec{V}_T$ relative to warm and cold air in the Northern (Southern) hemisphere?

Cold air advection (CAA) – the wind blows from a region of cooler temperatures to a region of warmer temperatures



Warm air advection (WAA) – the wind blows from a region of warmer temperatures to a region of cooler temperatures



Backing – wind turns counterclockwise with height

Veering – wind turns clockwise with height

In what direction does the geostrophic wind turn for the warm advection (cold advection) case?

Using this information we can:

- estimate the layer averaged temperature advection based on a single wind profile

- estimate the geostrophic wind at upper levels based on the geostrophic wind and temperature field at a lower level

Example: Determine whether CAA or WAA is occurring using a radiosonde observed wind profile

Barotropic and Baroclinic Atmospheres

Barotropic – density depends only on pressure [$\rho = \rho(p)$]

From the ideal gas law: $\rho = \dfrac{p}{RT}$

We find for a barotropic atmosphere that:

$T = T(p)$,

$\nabla_p T = 0$ (i.e. temperature is constant on constant pressure surfaces),

$\dfrac{\partial \langle T \rangle}{\partial x} = \dfrac{\partial \langle T \rangle}{\partial y} = 0$, and

$\vec{V}_T = 0$

The geostrophic wind does not vary with height in a barotropic atmosphere.

Baroclinic – density depends on both temperature and pressure

For a baroclinic atmoshphere the geostrophic wind can vary with height.

Vertical Motion

What is the relationship between ω and w?

$$\omega \equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p + w \frac{\partial p}{\partial z},$$

where:

$\vec{V} = \vec{V}_g + \vec{V}_a$ is the horizontal wind

and $\vec{V}_a << \vec{V}_g$ ($\vec{V}_a$ ~ 10% $\vec{V}_g$) for mid-latitude weather systems

Noting that:

$\vec{V} \cdot \nabla p = \vec{V}_g \cdot \nabla p + \vec{V}_a \cdot \nabla p$,

but:

$\vec{V}_g = \frac{1}{\rho f} \vec{k} \times \nabla p$, so

$\vec{V}_g \cdot \nabla p = 0$ and

$\vec{V} \cdot \nabla p = \vec{V}_a \cdot \nabla p$

Then:

$$\omega = \frac{\partial p}{\partial t} + \vec{V}_a \cdot \nabla p + w \frac{\partial p}{\partial z},$$

$$\omega = \frac{\partial p}{\partial t} + \vec{V}_a \cdot \nabla p - w \rho g$$

where we have used the hydrostatic equation to replace $\frac{\partial p}{\partial z}$

Scale analysis of this equation for mid-latitude weather systems gives:

$\frac{\partial p}{\partial t}$ ~ 10 mb day^{-1}

$\vec{V}_a \cdot \nabla p$ ~ (1 m s^{-1})(1 Pa km^{-1}) ~ 1 mb day $^{-1}$

$w\rho g$ ~ 100 mb day^{-1}

So:

$$\omega \approx -w\rho g$$

$$w \approx -\frac{\omega}{\rho g}$$

As noted in Chapter 2 *w* is of the order of 1 cm s^{-1} (10^{-2} m s^{-1}) for mid-latitude weather systems and is difficult measure accurately.

w can be estimated using the continuity equation (kinematic method) or the thermodynamic energy equation (adiabatic method).

Kinematic Method

For the kinematic method of estimating *w* we will integrate the continuity equation, in isobaric coordinates, between pressure levels p_s and p.

$$\int_{p_s}^{p} \partial\omega = -\int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p \partial p$$

$$\omega(p) = \omega(p_s) - \left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)_p (p - p_s),$$

where $\langle \ \rangle$ indicates a pressure weighted vertical average

Using $\omega \approx -w\rho g$ and *z* is the height of pressure level *p* and z_s is the height of pressure level p_s gives:

$$-\rho(z)gw(z) = -\rho(z_s)gw(z_s) + (p_s - p)\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)_p$$

$$w(z) = \frac{\rho(z_s)w(z_s)}{\rho(z)} - \frac{(p_s - p)}{\rho(z)g}\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)_p$$

In order to estimate *w*(*z*) we will need to estimate the value of the horizontal divergence of the wind:

$$\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)_p = \nabla_p \cdot \vec{V}$$

Recall that $\nabla_p \cdot \vec{V}_g = 0$ so $\nabla_p \cdot \vec{V} = \nabla_p \cdot \vec{V}_a$, but $|\vec{V}_a| \sim 0.1|\vec{V}_g|$.

This means that even a small error in the observed wind will result in a large relative error in $\vec{V}_a$ and thus $\nabla_p \cdot \vec{V}_a$ is difficult to calculate accurately.

Adiabatic Method

If we assume that the diabatic heating rate (*J*) is small then the thermodynamic energy equation reduces to:

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = 0$$

and

$$\omega = \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

$$-\rho g w = \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

$$w = -\frac{1}{\rho g S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

We can use this equation to estimate *w* based on the local rate of temperature change $\left(\frac{\partial T}{\partial t}\right)$, horizontal temperature advection $\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$, and the stability ($S_p$).

This equation can be difficult to use over a large area, as $\frac{\partial T}{\partial t}$ can be difficult to estimate.

This equation will not give reasonable results in areas of large diabatic heating (such as in clouds).

Another method of estimating ω and vertical velocity is discussed in Chapter 6.

Surface Pressure Tendency

Changes in surface pressure (p_s) can be used to indicate the approach of low pressure systems.

Using the continuity equation in isobaric coordinates we can derive an expression for the surface pressure tendency $\left(\frac{\partial p_s}{\partial t}\right)$.

Integrating the continuity equation from the surface to the top of the atmosphere (p = 0) and assuming that $\omega(0) = 0$ gives:

$$\omega(p_s) = \int_{p_s}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p \partial p = -\int_{0}^{p_s} (\nabla \cdot V)\partial p$$

Using:

$$\omega(p_s) \equiv \frac{Dp_s}{Dt} = \frac{\partial p_s}{\partial t} + \vec{V} \cdot \nabla p + w\frac{\partial p}{\partial z}$$

and noting that:

$w = 0$ over flat ground,

$\vec{V} \cdot \nabla p = \vec{V}_g \cdot \nabla p + \vec{V}_a \cdot \nabla p$,

$\vec{V}_g \cdot \nabla p = 0$, and

$\vec{V}_a \cdot \nabla p \approx 0$

gives:

$$\frac{\partial p_s}{\partial t} \approx -\int_{0}^{p_s} (\nabla \cdot \vec{V})\partial p$$

This equation indicates that the rate of surface pressure change is equal to the total convergence $(-\nabla \cdot \vec{V})$ into the column above the surface.

Consider a flow with only a zonal component:

What will cause $-\nabla \cdot \vec{V}$ to be positive (negative)?

What impact will this have on the surface pressure?

In practice it is difficult to use this equation to predict changes in the surface pressure since:

- it is difficult to accurately calculate $-\nabla \cdot \vec{V}$ from observations

- it is often observed that regions of convergence at one level in the atmosphere are often nearly compensated for by areas of divergence at other levels.

This equation can still be used to provide a conceptual understanding of the mechanisms that can cause surface pressure to vary.

Example: Development of a thermal low