Basic Conservation Laws

Atmospheric motions are governed by three principals:

- conservation of momentum
- conservation of mass
- conservation of energy

These conservation laws can be applied to a control volume of the atmosphere at a fixed location (Eulerian) or to a control volume of the atmosphere that is moving with the flow (Lagrangian)

What are the independent variables for the atmospheric field variables in Eulerian and Lagrangian frames of reference?

Total Differentiation

Total, substantial, or material derivative: $\frac{D}{Dt}$

This derivative is the time derivative following the motion (Lagrangian)

It will often be easiest to derive our conservation laws in a Lagrangian frame.

Local or partial derivative: $\frac{\partial}{\partial t}$

This derivative is the time derivative at a fixed location (Eulerian)

How can we relate the time derivative in a Lagrangian frame to the time derivative in an Eulerian frame?

Consider the field variable temperature [$T(x,y,z,t)$]

The position of an air parcel is given by (x,y,z) and is a function of time so,

$x = x(t)$, $y = y(t)$, and $z = z(t)$

Following the air parcel the rate of change of temperature is given by $\frac{DT}{Dt}$

We can relate the total change in temperature (δT) to changes at a fixed location and as a function of position:

$$\delta T = \left(\frac{\partial T}{\partial t}\right)\delta t + \left(\frac{\partial T}{\partial x}\right)\delta x + \left(\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial T}{\partial z}\right)\delta z + \text{ higher order terms}$$

Dividing by δt and taking the limit as δt goes to zero gives:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{Dx}{Dt} + \frac{\partial T}{\partial y}\frac{Dy}{Dt} + \frac{\partial T}{\partial z}\frac{Dz}{Dt}$$

where $\lim_{\delta t \to 0} \frac{\delta T}{\delta t} \equiv \frac{DT}{Dt}$

Also: $\frac{Dx}{Dt} \equiv u$, $\frac{Dy}{Dt} \equiv v$, $\frac{Dz}{Dt} \equiv w$, so

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

or

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{U} \cdot \nabla T\text{, where}$$

$\vec{U} = u\vec{i} + v\vec{j} + w\vec{k}$ and $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$

The local rate of temperature change is thus given by:

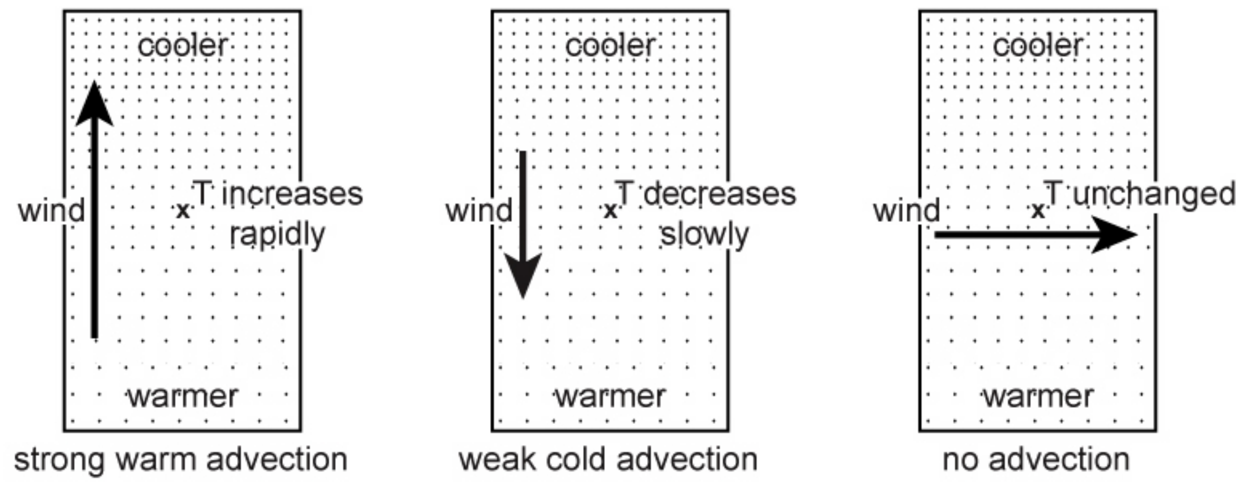
$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{U} \cdot \nabla T$$

What does each term in this equation represent?

Advection – the change in properties at a fixed location due to the replacement of the original air parcel at that location with a new air parcel with different properties

Temperature advection: $-\vec{U} \cdot \nabla T$

When will $-\vec{U} \cdot \nabla T$ be positive and negative?



Warm air advection – warmer air is replacing cooler air at a given location
Cold air advection – cooler air is replacing warmer air at a given location

What determines the magnitude of the temperature advection?

Example: Calculate temperature advection from a surface weather map

We can write similar equations for other variables:

$$\frac{\partial \theta}{\partial t} = \frac{D\theta}{Dt} - \vec{U} \cdot \nabla \theta$$

For a dry adiabatic process what is the value of $\frac{D\theta}{Dt}$?

In this case what can cause θ to vary at a fixed location?

Total Differentiation of a Vector in a Rotating Coordinate System

Newton's second law can be used to derive an equation that describes conservation of momentum (one of the basic principles of atmospheric dynamics), but this law applies to motion in an inertial reference frame.

In order to apply this law in a non-inertial reference frame we either need to consider apparent forces that arise due to the motion of the non-inertial reference frame or we need to relate the acceleration vector in an inertial reference frame to the acceleration vector in a non-inertial reference frame.

Consider vector $\vec{A}$:

$\vec{A} = A'_x \vec{i}' + A'_y \vec{j}' + A'_z \vec{k}'$ in an inertial reference frame

$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$ in a reference frame rotating with angular velocity Ω.

Let $\dfrac{D_a \vec{A}}{Dt}$ be the total derivative in the inertial reference frame, then

$$\frac{D_a \vec{A}}{Dt} = \frac{DA'_x}{Dt}\vec{i}' + \frac{DA'_y}{Dt}\vec{j}' + \frac{DA'_z}{Dt}\vec{k}'$$

This can also be written in terms of the components of $\vec{A}$ in the non-inertial reference frame as:

$$\frac{D_a \vec{A}}{Dt} = \frac{DA_x}{Dt}\vec{i} + A_x\frac{D_a \vec{i}}{Dt} + \frac{DA_y}{Dt}\vec{j} + A_y\frac{D_a \vec{j}}{Dt} + \frac{DA_z}{Dt}\vec{k} + A_z\frac{D_a \vec{k}}{Dt}$$

What do the terms $A_x\dfrac{D_a \vec{i}}{Dt}, A_y\dfrac{D_a \vec{j}}{Dt}, A_z\dfrac{D_a \vec{k}}{Dt}$ represent?

The total derivative in the non-inertial reference frame is given by:

$$\frac{D\vec{A}}{Dt} \equiv \frac{DA_x}{Dt}\vec{i} + \frac{DA_y}{Dt}\vec{j} + \frac{DA_z}{Dt}\vec{k}$$

In the rotating reference frame the change of $\vec{i}$ ($\delta\vec{i}$) is given by:

$$\delta\vec{i} = \frac{\partial\vec{i}}{\partial\lambda}\delta\lambda + \frac{\partial\vec{i}}{\partial\phi}\delta\phi + \frac{\partial\vec{i}}{\partial z}\delta z$$

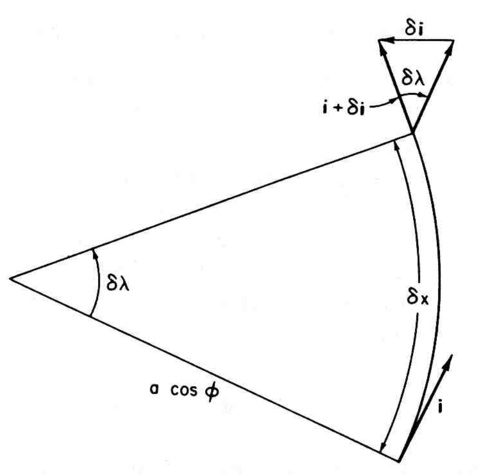
This expression relates $\delta\vec{i}$ to changes in longitude (λ), latitude (ϕ), and height (z).

For solid body rotation $\delta\lambda = \Omega\delta t$, $\delta\phi$ = 0, and δz = 0, so

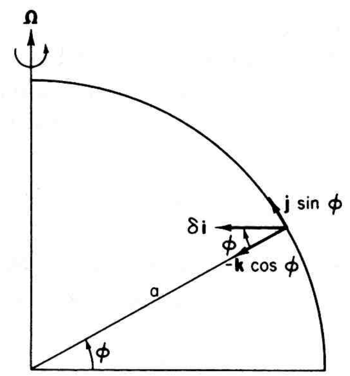
$$\delta\vec{i} = \frac{\partial\vec{i}}{\partial\lambda}\delta\lambda$$

Dividing by δt, taking the limit as $\delta t \to 0$, and noting that $\Omega = \frac{\partial\lambda}{\partial t}$ gives:

$$\frac{\delta\vec{i}}{\delta t} = \frac{\partial\vec{i}}{\partial\lambda}\frac{\delta\lambda}{\delta t}$$
$$\frac{D_a\vec{i}}{Dt} = \frac{\partial\vec{i}}{\partial\lambda}\frac{\partial\lambda}{\partial t}$$
$$\frac{D_a\vec{i}}{Dt} = \Omega\frac{\partial\vec{i}}{\partial\lambda}$$



From this figure we note that the direction of $\frac{D_a\vec{i}}{Dt}$ is towards the center of rotation as $\delta\lambda \to 0$



As shown in this figure the vector $\frac{D_a \vec{i}}{Dt}$ has components parallel to unit vectors $\vec{j}$ and $\vec{k}$.

Noting that $\frac{D_a \vec{i}}{Dt}$ has a magnitude equal to Ω gives:

$$\frac{D_a \vec{i}}{Dt} = \Omega \vec{j} \sin\phi - \Omega \vec{k} \cos\phi$$

Since $\vec{\Omega} = 0\vec{i} + \Omega \vec{j} \cos\phi + \Omega \vec{k} \sin\phi$:

$$\frac{D_a \vec{i}}{Dt} = \vec{\Omega} \times \vec{i}$$

Similarly $\frac{D_a \vec{j}}{Dt} = \vec{\Omega} \times \vec{j}$ and $\frac{D_a \vec{k}}{Dt} = \vec{\Omega} \times \vec{k}$

This then gives:

$$\frac{D_a \vec{A}}{Dt} = \frac{DA_x}{Dt}\vec{i} + A_x \frac{D_a \vec{i}}{Dt} + \frac{DA_y}{Dt}\vec{j} + A_y \frac{D_a \vec{j}}{Dt} + \frac{DA_z}{Dt}\vec{k} + A_z \frac{D_a \vec{k}}{Dt}$$

$$\frac{D_a \vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + \vec{\Omega} \times \vec{A}$$

Vectorial Form of the Momentum Equation in Rotating Coordinates

From Newton's second law $\vec{a} = \sum \frac{\vec{F}}{m}$.

Applied in an inertial reference frame this gives:

$$\vec{a} = \frac{D_a \vec{U}_a}{Dt} = \sum \frac{\vec{F}}{m}$$

As seen in Chapter 1 we needed to include apparent forces in this expression when applying Newton's second law in a non-inertial reference frame.

We can arrive at the same result by using the relationship between $\frac{D_a \vec{A}}{Dt}$ and $\frac{D\vec{A}}{Dt}$ derived above:

$$\frac{D_a \vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + \vec{\Omega} \times \vec{A}$$

First, apply this relationship to the position vector $\vec{r}$ to get:

$$\frac{D_a \vec{r}}{Dt} = \frac{D\vec{r}}{Dt} + \vec{\Omega} \times \vec{r}$$
$$\vec{U}_a = \vec{U} + \vec{\Omega} \times \vec{r}$$

What does each term above represent?

We can also apply the relationship between $\frac{D_a \vec{A}}{Dt}$ and $\frac{D\vec{A}}{Dt}$ to $\vec{U}_a$ to get:

$$\frac{D_a \vec{U}_a}{Dt} = \frac{D\vec{U}_a}{Dt} + \vec{\Omega} \times \vec{U}_a$$

Replacing $\vec{U}_a$ with $\vec{U}_a = \vec{U} + \vec{\Omega} \times \vec{r}$ on the RHS of this equation gives:

$$\frac{D_a \vec{U}_a}{Dt} = \frac{D\left(\vec{U} + \vec{\Omega} \times \vec{r}\right)}{Dt} + \vec{\Omega} \times \left(\vec{U} + \vec{\Omega} \times \vec{r}\right)$$
$$= \frac{D\vec{U}}{Dt} + \frac{D\left(\vec{\Omega} \times \vec{r}\right)}{Dt} + \vec{\Omega} \times \vec{U} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$
$$= \frac{D\vec{U}}{Dt} + \vec{\Omega} \times \frac{D\vec{r}}{Dt} + \vec{\Omega} \times \vec{U} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$
$$= \frac{D\vec{U}}{Dt} + \vec{\Omega} \times \vec{U} + \vec{\Omega} \times \vec{U} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$
$$\frac{D_a \vec{U}_a}{Dt} = \frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R}$$

where we have used the vector identity: $\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right) = -\Omega^2 \vec{R}$

What do each of the terms in this equation represent?

Using this with Newton's second law gives:

$$\frac{D_a \vec{U}_a}{Dt} = \frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R} = \sum \frac{\vec{F}}{m} = \vec{P}_g + \vec{g}^* + \vec{F}_r$$
$$\frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R} = \vec{P}_g + \vec{g}^* + \vec{F}_r$$

What do each of the terms in this equation represent?

Rearranging terms gives:

$$\frac{D\vec{U}}{Dt} = -2\vec{\Omega} \times \vec{U} + \vec{P}_g + \vec{g} + \vec{F}_r$$
$$\frac{D\vec{U}}{Dt} = -2\vec{\Omega} \times \vec{U} + \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

This equation is the momentum equation and represents the conservation of momentum in the atmosphere in a rotating reference frame.

Component Equations in Spherical Coordinates

It is often preferable to work with the equation of motion in component form rather than in vectorial form.

The coordinate system that is typically used is a spherical coordinate system with axes given by longitude (λ), latitude (ϕ), and vertical distance above sea level (*z*).

The unit vectors in this coordinate system point towards the east ($\vec{i}$), north ($\vec{j}$) and up ($\vec{k}$).

The three dimensional wind vector is given by:

$\vec{U} = u\vec{i} + v\vec{j} + w\vec{k}$, with $u \equiv \dfrac{Dx}{Dt}, v \equiv \dfrac{Dy}{Dt}, w \equiv \dfrac{Dz}{Dt}$

The distances *(Dx* and *Dy*) can be expressed as:

$$Dx = a\cos\phi D\lambda \qquad\qquad Dy = aD\phi$$

Where *a* = radius of the Earth

The direction of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ in this coordinate system are not constant and vary with position.

The acceleration of $\vec{U}$ is then given by:

$$\frac{D\vec{U}}{Dt} = \frac{Du}{Dt}\vec{i} + \frac{Dv}{Dt}\vec{j} + \frac{Dw}{Dt}\vec{k} + u\frac{D\vec{i}}{Dt} + v\frac{D\vec{j}}{Dt} + w\frac{D\vec{k}}{Dt}$$

The terms $u\dfrac{D\vec{i}}{Dt}, v\dfrac{D\vec{j}}{Dt}, w\dfrac{D\vec{k}}{Dt}$ represent the change in direction of the unit vectors as the air parcel moves and are given by:

$$\frac{D\vec{i}}{Dt} = \frac{u}{a\cos\phi}\left(\vec{j}\sin\phi - \vec{k}\cos\phi\right)$$

$$\frac{D\vec{j}}{Dt} = -\frac{u\tan\phi}{a}\vec{i} - \frac{v}{a}\vec{k}$$

$$\frac{D\vec{k}}{Dt} = \frac{u}{a}\vec{i} + \frac{v}{a}\vec{j}$$

The acceleration can be written in component form as:

$$\frac{D\vec{U}}{Dt} = \left(\frac{Du}{Dt} - \frac{uv}{a}\tan\phi + \frac{uw}{a}\right)\vec{i} + \left(\frac{Dv}{Dt} + \frac{u^2}{a}\tan\phi + \frac{vw}{a}\right)\vec{j} + \left(\frac{Dw}{Dt} - \frac{u^2 + v^2}{a}\right)\vec{k}$$

The forces acting on the air can be written in component form as:

Coriolis force:

$$-2\vec{\Omega} \times \vec{U} = -(2\Omega w\cos\phi - 2\Omega v\sin\phi)\vec{i} - (2\Omega u\sin\phi)\vec{j} + (2\Omega u\cos\phi)\vec{k}$$

Pressure gradient force:

$$P_g = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Gravity:

$$\vec{g} = -g\vec{k}$$

Friction (viscous) force:

$$\vec{F}_r = F_{rx}\vec{i} + F_{ry}\vec{j} + F_{rz}\vec{k}$$

The component form of the momentum equation is given by:

$$\frac{Du}{Dt} - \frac{uv}{a}\tan\phi + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{Dv}{Dt} - \frac{u^2}{a}\tan\phi + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz}$$

Curvature terms –terms that arise due to the curvature of the Earth. These terms are proportional to $1/a$

These terms take into account the changing direction of the unit vectors as the air moves through the spherical coordinate system.

These terms are nonlinear (i.e. they include products of the dependent variables) and make it difficult to handle these equations for theoretical analysis.

What other terms in these equations are nonlinear?

Scale Analysis of the Equations of Motion

Scale analysis allows us to estimate the magnitude of the terms in our equations and determine which terms may be neglected.

Not only can this simplify our equations by removing unimportant terms, but it can also allow us to filter certain types of motion from the equations.

The typical magnitude of the dependent variables can be specified based on observations.

For mid-latitude weather systems we find:

Scale	**Symbol**	**Magnitude**
Horizontal wind scale	U	10 ms^{-1}
Vertical wind scale	W	$10^{-2}ms^{-1}$
Horizontal length scale	L	10^6 m
Vertical length scale (depth of troposphere)	H	10^4 m
Time scale (L/U)	T	10^5 s
Kinematic viscosity	ν	10^{-5} m^2s^{-1}
Dynamic pressure scale	$\delta p/\rho$	10^3 m^2s^{-2}
Total pressure scale	P/ρ	10^5 m^2s^{-2}
Gravity	g	10 ms^{-2}
Density variation scale	$\delta\rho/\rho$	10^{-2}

The variation in pressure (δp) is normalized by density (ρ) such that $\delta p/\rho$ is approximately constant with height in the atmosphere despite the large change in p and ρ in the vertical direction.

The time scale (L/U) is an advective time scale and represents the time required for a weather system to move distance L assuming that the system is moving at the same speed as the wind (U).

The vertical wind scale (W) is difficult to measure for mid-latitude weather systems, but can be estimated based on the horizontal winds.

For mid-latitudes ϕ = 45º and $f_0 = 2\Omega\sin\phi = 2\Omega\cos\phi \sim 10^{-4}$ s^{-1}

Applying this scaling to the horizontal components of the equations of motion gives:

	A	B	C	D	E	F	G
$x-$Eq.	$\frac{Du}{Dt}$	$-2\Omega v\sin\phi$	$+2\Omega w\cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
$y-$Eq.	$\frac{Dv}{Dt}$	$+2\Omega u\sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	f_0U	f_0W	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
$(\mathrm{m\,s^{-2}})$	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

The friction terms are many orders of magnitude smaller than all of the other terms in the equations and can be neglected with minimal error.

The two largest terms are the pressure gradient force and the Coriolis force.

These terms are in approximate balance, within an error of ~10%.

Geostrophic relationship

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad\qquad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

This relationship is a diagnostic relationship – it cannot be used to predict changes over time (i.e. it is not a prognostic relationship)

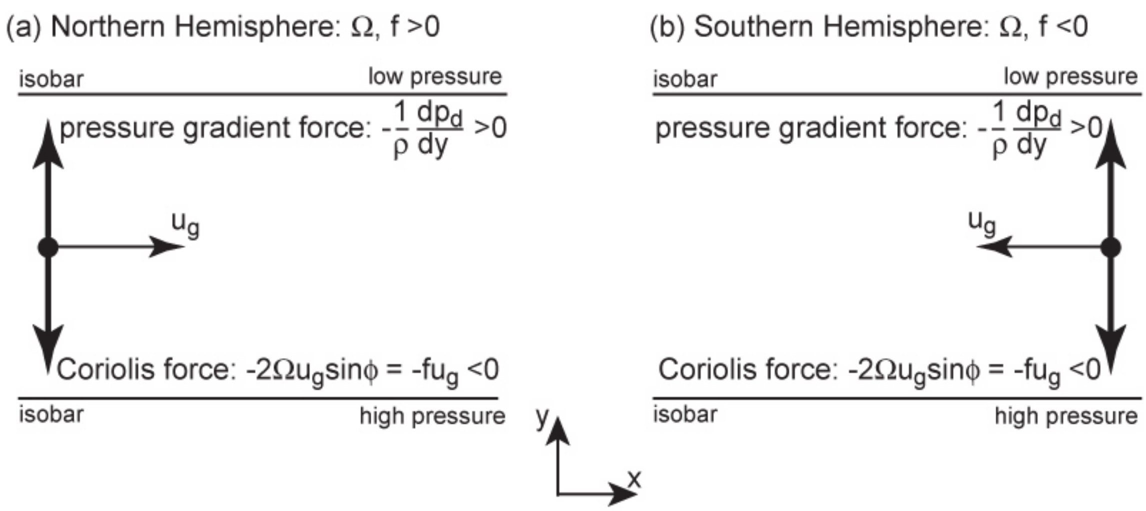
Geostrophic wind ($\vec{V}_g$) – the wind that exactly satisfies the geostrophic relationship

$$\begin{aligned}\vec{V}_g &= u_g\vec{i} + v_g\vec{j}\\ &\equiv \vec{k}\times\frac{1}{\rho f}\nabla p\\ &= -\frac{1}{\rho f}\frac{\partial p}{\partial y}\vec{i} + \frac{1}{\rho f}\frac{\partial p}{\partial x}\vec{j}\end{aligned}$$

Note that $\vec{V}_g$ can be evaluated for any pressure field, but will most closely approximate the actual wind for mid-latitude synoptic weather systems.

In this case $\vec{V}_g$ will be within 10-15% of the actual wind.

What is the direction of the geostrophic wind relative to the pressure field?



Since upper air weather maps are often presented as constant pressure maps it is useful to rewrite the equations for the geostrophic wind expressed on a constant pressure surface.

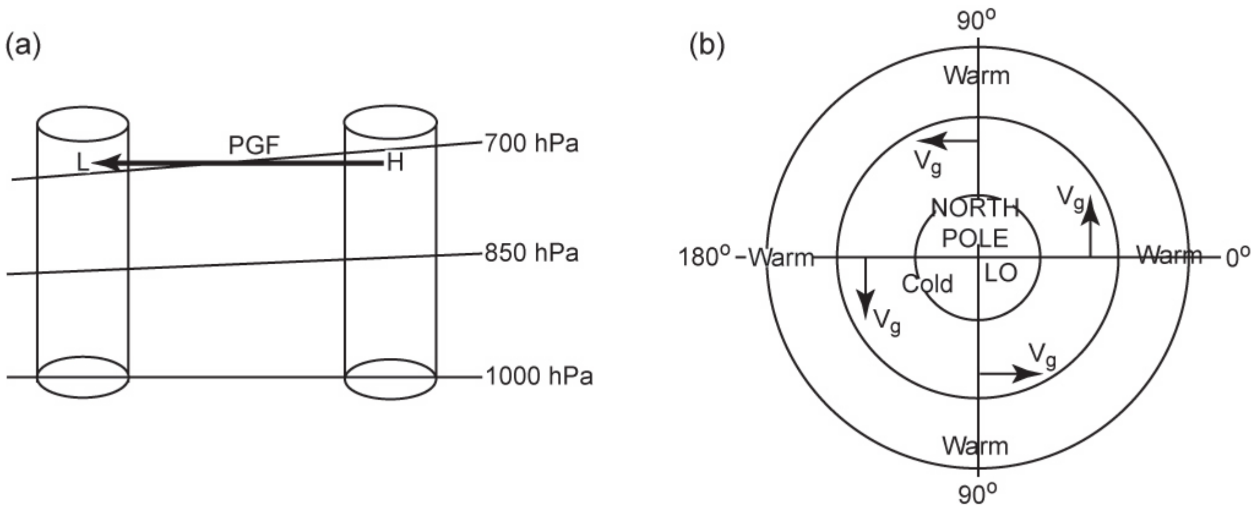
$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}\bigg|_p$$

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}\bigg|_p$$

Example: Calculation of the geostrophic wind from a weather map

Why do mid-latitude weather systems tend to move from west to east?

Why does the wind blow in a counterclockwise direction around areas of low pressure in the Northern hemisphere?



Approximate Prognostic Equations – the Rossby number

In order for the momentum equations to be used as prognostic equations we must retain the acceleration term (D/Dt).

This gives:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right) \qquad \frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

The acceleration term is one order of magnitude smaller than the pressure gradient force or the Coriolis force.

The ratio of the magnitude of the acceleration term ($Du/Dt \sim U^2/L$) to the magnitude of the Coriolis force ($\sim f_0 U$) is known as the Rossby number:

$$Ro \equiv \frac{U}{f_0 L}$$

The geostrophic approximation is most valid for small Ro (~0.1)

Hydrostatic approximation

Scale analysis of the vertical momentum equation gives:

z - Eq.	Dw/Dt	$-2\Omega u\cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

The two largest terms are the vertical pressure gradient force and gravity.

All other terms are at least 3 orders of magnitude smaller, and thus the balance between the vertical pressure gradient force and gravity (the hydrostatic approximation) is accurate to $O(10^{-3})$.

This gives:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

It is also necessary to show that the horizontally varying pressure is also in hydrostatic balance.

To do this we will define a standard pressure [$p_0(z)$], which is the horizontally averaged pressure at each height and a standard density [$\rho_0(z)$], such that $p_0(z)$ and $\rho_0(z)$ are in hydrostatic balance.

$$\frac{1}{\rho_0}\frac{dp_0}{dz} = -g$$

With these definitions the total pressure and density are given by:

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$
$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

Substituting this into the hydrostatic approximation $\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$ gives:

$$-\frac{1}{\rho_0 + \rho'}\frac{\partial(p_0 + p')}{\partial z} - g = 0$$

For ρ'/ρ_0 << 1 $\frac{1}{\rho_0 + \rho'} \cong \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)$. This gives:

$$-\frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)\frac{\partial(p_0 + p')}{\partial z} - g = 0$$

$$-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0{}^2}\frac{\partial p_0}{\partial z} + \frac{\rho'}{\rho_0{}^2}\frac{\partial p'}{\partial z} - g = 0$$

The sum of the first and last terms on the LHS of this equation is zero from the hydrostatic approximation for the standard state, leaving:

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0{}^2}\frac{\partial p_0}{\partial z} + \frac{\rho'}{\rho_0{}^2}\frac{\partial p'}{\partial z} = 0$$

Scale analysis of this equation indicates that the last term on the LHS of this equation is two orders of magnitude smaller than the other terms, so:

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0{}^2}\frac{\partial p_0}{\partial z} = 0$$

Using $\frac{1}{\rho_0}\frac{dp_0}{dz} = -g$ to substitute into the last term on the LHS gives:

$$-\frac{1}{\rho_0}\left(\frac{\partial p'}{\partial z} + \rho' g\right) = 0$$

,

$$\frac{\partial p'}{\partial z} + \rho' g = 0$$

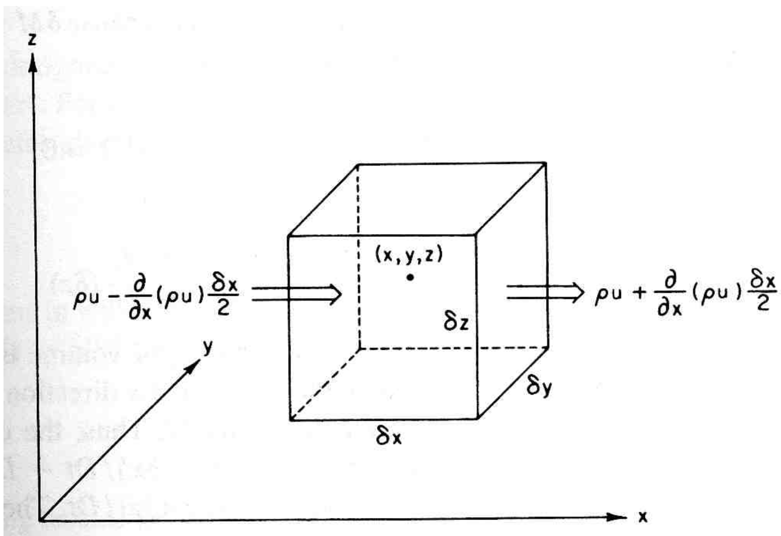
which indicates that the horizontally varying pressure is also in the approximate hydrostatic balance.

What does the approximate hydrostatic balance that exists in mid-latitude weather system imply about our ability to use the vertical momentum equation to predict changes in the vertical velocity?

The Continuity Equation

The second principal governing atmospheric motions is conservation of mass, which is expressed by the continuity equation.

Eulerian derivation



Consider a volume ($\delta x \delta y \delta z$) that is fixed in space.

Atmospheric mass can flow into and out of this volume due to the wind.

From this figure the mass flux into the left face is given by:

$$\left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z$$

and the mass flux out of the right face is given by:

$$\left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z$$

The rate of mass change in the volume due to these fluxes is:

$$\frac{\partial M}{\partial t} = \left\{ \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \right] - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \right] \right\} \delta y \delta z$$

$$\frac{\partial M}{\partial t} = -\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$$

Considering all three faces of this volume gives:

$$\frac{\partial M}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z$$

Dividing by the volume ($\delta x \delta y \delta z$) gives the rate of change of density:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = -\nabla \cdot \left(\rho \vec{U} \right)$$

What conditions will cause $-\nabla \cdot \left(\rho \vec{U} \right)$ to be positive or negative?

This equation is referred to as the mass divergence form of the continuity equation, and states that the local rate of change of density is equal to minus the mass divergence.

This equation can also be expressed as:

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{U} - \vec{U} \cdot \nabla \rho$$

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{U}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \vec{U}$$

This is the velocity divergence form of the continuity equation, and states that the fractional rate of density change following the motion is equal to minus the velocity divergence.

When will $-\nabla \cdot \vec{U}$ (the velocity divergence) be positive or negative?

How does the volume of our control box change for each of these cases?

Lagrangian Derivation of the Continuity Equation

This derivation is an alternate method of deriving the velocity divergence form of the continuity equation.

Consider a fixed mass of air ($\delta M = \rho \delta V = \rho \delta x \delta y \delta z$).

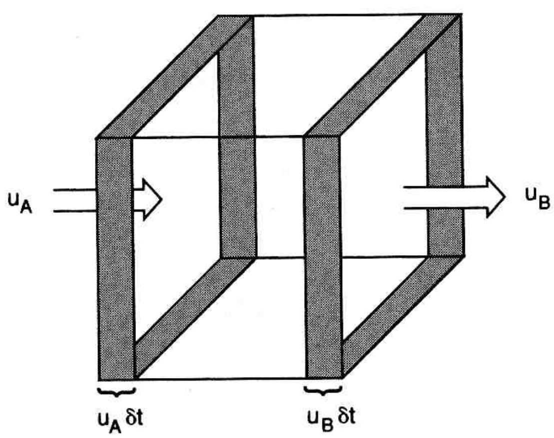
Since δM is constant $\rho \delta x \delta y \delta z$ is also constant and changes in ρ are balanced by changes in $\delta x \delta y \delta z$.

$$\frac{1}{\delta M}\frac{D(\delta M)}{Dt} = \frac{1}{\rho \delta V}\frac{D(\rho \delta V)}{Dt} = \frac{1}{\delta V}\frac{D(\delta V)}{Dt} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

Noting that:

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \frac{1}{\delta x \delta y \delta z}\frac{D(\delta x \delta y \delta z)}{Dt} = \frac{1}{\delta x}\frac{D(\delta x)}{Dt} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta z}\frac{D(\delta z)}{Dt}$$



The faces of the box move at the speed of the wind.

$$u_A = \frac{Dx}{Dt} \qquad u_B = \frac{D(x + \delta x)}{Dt}$$

$$\delta u = u_B - u_A = \frac{D(x + \delta x)}{Dt} - \frac{Dx}{Dt} = \frac{D(\delta x)}{Dt}$$

Similarly, $\delta v = \frac{D(\delta y)}{Dt}$ and $\delta w = \frac{D(\delta z)}{Dt}$

This gives: $\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \frac{1}{\delta x \delta y \delta z}\frac{D(\delta x \delta y \delta z)}{Dt} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$

In the limit $\delta V \to 0$:

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{U}$$
,
$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \vec{U}$$

which is the velocity divergence form of the continuity equation.

Scale Analysis of the Continuity Equation

We will use the same method as was applied for the scale analysis of the hydrostatic equation:

Use $\rho = \rho_0(z) + \rho'(x,y,z,t)$

Note that $\frac{\rho'}{\rho_0} \sim 10^{-2} << 1$ such that $1 - \frac{\rho'}{\rho_0} \approx 1$ and $\frac{1}{\rho_0 + \rho'} \cong \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right) \cong \frac{1}{\rho_0}$

Substituting this into the velocity divergence form of the continuity equation gives:

$$\frac{1}{\rho_0}\left[\frac{D\rho_0}{Dt} + \frac{D\rho'}{Dt}\right] + \nabla \cdot \vec{U} = 0$$

Since $\rho_0 = \rho_0(z)$:

$$\frac{D\rho_0}{Dt} = \frac{\partial \rho_0}{\partial t} + u\frac{\partial \rho_0}{\partial x} + v\frac{\partial \rho_0}{\partial y} + w\frac{\partial \rho_0}{\partial z} = w\frac{d\rho_0}{dz}$$

The expanded velocity divergence form of the equation is then:

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t}+u\frac{\partial \rho'}{\partial x}+v\frac{\partial \rho'}{\partial y}+w\frac{\partial \rho'}{\partial z}\right]+\frac{w}{\rho_0}\frac{d\rho_0}{dz}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Scale analysis of this equation indicates:

$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t}\sim\frac{\delta\rho}{\rho}\frac{1}{T}\sim\frac{\rho'}{\rho_0}\frac{U}{L}\sim 10^{-2}\frac{10\text{ m s}^{-1}}{10^{6}\text{ m}}\sim 10^{-7}\text{ s}^{-1}$$

$$\frac{u}{\rho_0}\frac{\partial \rho'}{\partial x},\frac{v}{\rho_0}\frac{\partial \rho'}{\partial y}\sim\frac{U}{\rho}\frac{\delta\rho}{L}\sim\frac{\rho'}{\rho_0}\frac{U}{L}\sim 10^{-2}\frac{10\text{ m s}^{-1}}{10^{6}\text{ m}}\sim 10^{-7}\text{ s}^{-1}$$

$$\frac{w}{\rho_0}\frac{\partial \rho'}{\partial z}\sim\frac{W}{\rho}\frac{\delta\rho}{H}\sim\frac{\rho'}{\rho_0}\frac{W}{H}\sim 10^{-2}\frac{10^{-2}\text{ m s}^{-1}}{10^{4}\text{ m}}\sim 10^{-8}\text{ s}^{-1}$$

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz}\sim\frac{W}{\rho}\frac{\rho}{H}\sim\frac{\rho_0}{\rho_0}\frac{W}{H}\sim 1\frac{10^{-2}\text{ m s}^{-1}}{10^{4}\text{ m}}\sim 10^{-6}\text{ s}^{-1}$$

Why did we use $\frac{\rho}{H}$ instead of $\frac{\delta\rho}{H}$ to estimate the scale of $\frac{d\rho_0}{dz}$?

$$\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}\sim\frac{U}{L}\sim\frac{10\text{ m s}^{-1}}{10^{6}\text{ m}}\sim 10^{-5}\text{ s}^{-1},$$

but $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ tends to cancel so $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\sim 10^{-6}\text{ s}^{-1}$

$$\frac{\partial w}{\partial z}\sim\frac{W}{H}\sim\frac{10^{-2}\text{ m s}^{-1}}{10^{4}\text{ m}}\sim 10^{-6}\text{ s}^{-1}$$

Keeping the largest terms from this scale analysis gives:

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz}+\nabla\cdot\vec{U}=0$$

This can also be written as:

$$\nabla\cdot\left(\rho_0\vec{U}\right)=0$$

This indicates that for synoptic scale motions the mass flux calculated using the basic state density (ρ_0) is nondivergent.

i.e. changes in $\frac{\partial(\rho_0 u)}{\partial x},\frac{\partial(\rho_0 v)}{\partial y},\frac{\partial(\rho_0 w)}{\partial z}$ exactly balance

This is similar to the idea of an incompressible fluid, but for an incompressible fluid:

$\frac{D\rho}{Dt}=0$ (ρ does not change following the motion)

and the continuity equation $\left(\frac{1}{\rho}\frac{D\rho}{Dt}+\nabla\cdot\vec{U}=0\right)$ reduces to $\nabla\cdot\vec{U}=0$

This result indicates that in an incompressible fluid the flow is nondivergent and changes in $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial w}{\partial z}$ exactly balance.

In the atmosphere $\frac{D\rho}{Dt} \neq 0$ (ρ can change following the motion),

but for purely horizontal flow (*w* = 0) the synoptically scaled continuity equation reduces to:

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz} + \nabla \cdot \vec{U} = 0$$
$$\nabla \cdot \vec{U} = 0$$

and the horizontal flow is nondivergent (i.e. $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ exactly balance)

The compressibility of the atmosphere only needs to be considered when there is vertical motion, and then $\frac{d\rho_0}{dz}$ must be accounted for.

Thermodynamic Energy Equation

The third fundamental principal of atmospheric dynamics is the conservation of energy.

First Law of Thermodynamics – the heat added to a system is equal to the change in internal energy plus the work done by the system

This law applies to a system that is in thermodynamic equilibrium (i.e. a system at rest).

Can this law be applied to the atmosphere that is in motion?

We will consider a fixed mass of air that we will follow through the atmosphere (a Lagrangian control volume) as our thermodynamic system, but this system will not be in thermodynamic equilibrium because it is in motion.

For this system the total energy is equal to the sum of the internal energy (the energy associated with molecular properties) and the kinetic energy due to the macroscopic motion (movement) of the system.

For this system the heat added (diabatic heating) is equal to the change in the total energy plus the work done by the system.

This can also be expressed as the diabatic heating rate (J) is equal to the rate of change of the total energy plus the rate at which work is done on the system.

Using:

e = internal energy per unit mass

$\frac{1}{2}\vec{U}\cdot\vec{U}$ = kinetic energy per unit mass

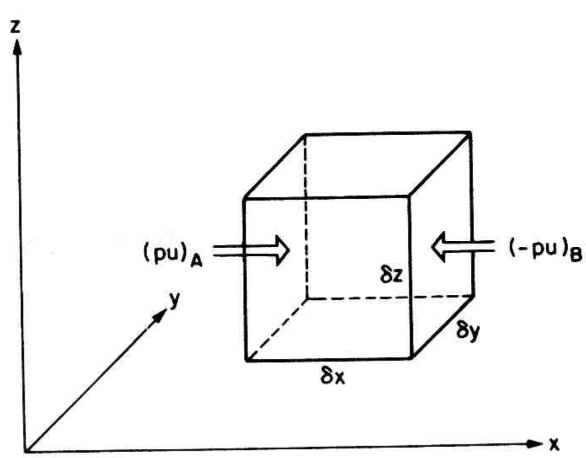
The total energy in the Lagrangian control volume, with density ρ and volume δV, is given by:

$$\rho\left[e+\frac{1}{2}\vec{U}\cdot\vec{U}\right]\delta V$$

The rate at which a force does work is equal to the dot product of the force and velocity vectors.

What forces act on the atmosphere?

Rate of work done by the pressure gradient force:



Rate of work = $\vec{F}\cdot\vec{U}$

Since $p = F/A$ then $F = pA$

The rate at which work is done by the pressure in the x-direction is given by: $pu\delta y\delta z$

For the control volume the rate at which pressure does work on the volume is given by:

$$(pu)_A \delta y \delta z - (pu)_B \delta y \delta z$$

Noting that $(pu)_B$ can be expressed as:

$$(pu)_B \approx (pu)_A + \frac{\partial (pu)}{\partial x} \delta x$$

The rate of work done by the pressure in the *x*-direction is:

$$-\frac{\partial (pu)}{\partial x} \delta V$$

Similarly the work done by the pressure in the *y*- and *z*-directions is:

$$-\frac{\partial (pv)}{\partial y} \delta V, -\frac{\partial (pw)}{\partial z} \delta V$$

and the total rate of work done by the pressure is:

$$-\nabla \cdot \left(p\vec{U} \right) \delta V$$

Rate of work done by the friction force:

We will neglect the work done by the friction force based on synoptic scaling arguments.

Rate of work done by the Coriolis force:

The Coriolis force is given by: $-2\vec{\Omega} \times \vec{U}$

This force is perpendicular to $\vec{U}$ and thus $\left(-2\vec{\Omega} \times \vec{U}\right) \cdot \vec{U} = 0$, so the Coriolis force does no work on the atmosphere.

Rate of work done by gravity:

$\rho\vec{g}\cdot\vec{U}\delta V$

Conservation of energy for the Lagrangian control volume gives:

$$\frac{D\left[\rho\left(e+\frac{1}{2}\vec{U}\cdot\vec{U}\right)\delta V\right]}{Dt}=-\nabla\cdot\left(p\vec{U}\right)\delta V+\rho\vec{g}\cdot\vec{U}\delta V+\rho J\delta V$$

Use of the chain rule on the term expressing the rate of total energy change gives:

$$\begin{aligned}\frac{D\left[\rho\left(e+\frac{1}{2}\vec{U}\cdot\vec{U}\right)\delta V\right]}{Dt}&=\rho\delta V\frac{D\left[e+\frac{1}{2}\vec{U}\cdot\vec{U}\right]}{Dt}+\left(e+\frac{1}{2}\vec{U}\cdot\vec{U}\right)\frac{D(\rho\delta V)}{Dt}\\&=\rho\delta V\frac{D\left[e+\frac{1}{2}\vec{U}\cdot\vec{U}\right]}{Dt}\end{aligned}$$

Dividing by δV and noting that $\vec{g}\cdot\vec{U}=-gw$ gives:

$$\rho\frac{D\left(e+\frac{1}{2}\vec{U}\cdot\vec{U}\right)}{Dt}=-\nabla\cdot\left(p\vec{U}\right)-\rho gw+\rho J$$

$$\rho\frac{De}{Dt}+\rho\frac{D\left(\frac{1}{2}\vec{U}\cdot\vec{U}\right)}{Dt}=-U\cdot\nabla p-p\nabla\cdot U-\rho gw+\rho J$$

Take the dot product of $\vec{U}$ with the momentum equation (and neglecting the friction term) gives:

$$\vec{U}\cdot\frac{D\vec{U}}{Dt}=\vec{U}\cdot\left(-2\vec{\Omega}\times\vec{U}\right)-\vec{U}\cdot\left(\frac{1}{\rho}\nabla p\right)+\vec{U}\cdot\vec{g}$$

$$\frac{D\left(\frac{1}{2}\vec{U}\cdot\vec{U}\right)}{Dt}=-\frac{1}{\rho}\vec{U}\cdot\nabla p-wg$$

$$\rho\frac{D\left(\frac{1}{2}\vec{U}\cdot\vec{U}\right)}{Dt}=-\vec{U}\cdot\nabla p-\rho wg$$

This equation represents the balance of mechanical energy due to the motion of the fluid (rate of change of kinetic energy following the motion).

Subtracting this from the conservation of energy equation gives:

$$\rho\frac{De}{Dt}=-p\nabla\cdot\vec{U}+\rho J$$

This equation represents the thermal energy balance (the rate of change of internal energy following the motion).

The mechanical energy balance can be rewritten using the definition of geopotential:

$$d\Phi = gdz$$

$$\frac{d\Phi}{dt}=g\frac{dz}{dt}=gw$$

$$\rho\frac{D\left(\frac{1}{2}\vec{U}\cdot\vec{U}\right)}{Dt}=-\vec{U}\cdot\nabla p-\rho\frac{d\Phi}{dt}$$

$$\rho\frac{D\left(\frac{1}{2}\vec{U}\cdot\vec{U}+\Phi\right)}{Dt}=-\vec{U}\cdot\nabla p$$

This equation is known as the mechanical energy equation.

The sum of the kinetic energy and the gravitational potential energy (geopotential) is the mechanical energy.

This equation states that the rate of change of mechanical energy, following the air parcel, is equal to the rate at which the pressure gradient force does work on the air parcel.

The thermal energy equation can be rewritten by noting that:

$$\frac{1}{\rho}\nabla \cdot \vec{U} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{D\alpha}{Dt},$$

where we have used the continuity equation to replace $\left(\nabla \cdot \vec{U}\right)$.

Then:

$$\rho\frac{De}{Dt} = -p\nabla \cdot \vec{U} + \rho J$$

$$\frac{De}{Dt} = -\frac{p}{\rho}\nabla \cdot \vec{U} + J$$

$$\frac{De}{Dt} = -p\frac{D\alpha}{Dt} + J$$

$$J = \frac{De}{Dt} + p\frac{D\alpha}{Dt}$$

Note that $e = c_v T$, so:

$$J = \frac{De}{Dt} + p\frac{D\alpha}{Dt}$$

$$J = \frac{D(c_v T)}{Dt} + p\frac{D\alpha}{Dt}$$

$$J = c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt}$$

Thermodynamics of the Dry Atmosphere

Comparison of thermodynamic equations from Holton and Hakim and Wallace and Hobbs:

Holton and Hakim	Wallace and Hobbs
$J = c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt}$	$dq = c_v dT + p d\alpha$
$J = c_p \frac{DT}{Dt} - \alpha\frac{Dp}{Dt}$	$dq = c_p dT - \alpha dp$
$\frac{Ds}{Dt} = \frac{J}{T} = c_p \frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt}$	$ds = \frac{dq}{T} = c_p \frac{dT}{T} - R\frac{dp}{p}$
$\frac{Ds}{Dt} = \frac{J}{T} = c_p \frac{D\ln\theta}{Dt}$	$ds = \frac{dq}{T} = c_p \frac{d\theta}{\theta}$

Scale Analysis of the Thermodynamic Energy Equation

$$c_p \frac{D\ln\theta}{Dt} = \frac{J}{T}$$

$$\frac{c_p}{\theta}\frac{D\theta}{Dt} = \frac{J}{T}$$

Define $\theta = \theta_0(z) + \theta'(x,y,z,t)$, where θ_0 is the basic state potential temperature and θ' is the deviation from the basic state.

We will assume that $\frac{\theta'}{\theta_0} << 1$, so $\theta' << \theta_0$ and $\frac{1}{\theta_0 + \theta'} \approx \frac{1}{\theta_0}$

This gives:

$$\frac{1}{\theta}\frac{D\theta}{Dt} = \frac{J}{c_p T}$$

$$\frac{1}{\theta_0 + \theta'}\frac{D(\theta_0 + \theta')}{Dt} = \frac{J}{c_p T}$$

$$\frac{1}{\theta_0}\left[\frac{D\theta_0}{Dt} + \frac{D\theta'}{Dt}\right] = \frac{J}{c_p T}$$

$$\frac{1}{\theta_0}\left[\frac{\partial\theta_0}{\partial t} + u\frac{\partial\theta_0}{\partial x} + v\frac{\partial\theta_0}{\partial y} + w\frac{\partial\theta_0}{\partial z} + \frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y} + w\frac{\partial\theta'}{\partial z}\right] = \frac{J}{c_p T}$$

Since θ_0 = θ_0(z) $\frac{\partial\theta_0}{\partial t}, \frac{\partial\theta_0}{\partial x}, \frac{\partial\theta_0}{\partial y} = 0$ and

$$\frac{1}{\theta_0}\left[w\frac{\partial\theta_0}{\partial z} + \frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y} + w\frac{\partial\theta'}{\partial z}\right] = \frac{J}{c_p T}$$

We will assume that $\frac{\partial\theta'}{\partial z} << \frac{\partial\theta_0}{\partial z}$, so we can neglect $w\frac{\partial\theta'}{\partial z}$:

$$\frac{1}{\theta_0}\left[\frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y}\right] + \frac{w}{\theta_0}\frac{d\theta_0}{dz} = \frac{J}{c_p T}$$

$$\frac{1}{\theta_0}\left[\frac{\partial\theta'}{\partial t} + u\frac{\partial\theta'}{\partial x} + v\frac{\partial\theta'}{\partial y}\right] + w\frac{d\ln\theta_0}{\partial z} = \frac{J}{c_p T}$$

In the absence of clouds/precipitation and away from the surface of the Earth $\frac{J}{c_p} \leq$ 1 deg C day^{-1} in the troposphere.

Why would the heating rate given above differ in the presence of clouds/precipitation or near the surface of the Earth?

θ' ~ 4 deg C away from the surface of the Earth

$$\frac{T}{\theta_0}\left[\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+v\frac{\partial \theta'}{\partial y}\right]\sim\frac{T}{\theta}\left[\frac{\theta'}{L/U}+U\frac{\theta'}{L}+U\frac{\theta'}{L}\right]\sim\frac{T}{\theta}U\frac{\theta'}{L}$$

$$\frac{T}{\theta}U\frac{\theta'}{L}\sim(1)\left(10\text{ m s}^{-1}\right)\frac{4\text{ deg C}}{10^6\text{ m}}\sim 4\text{ deg C day}^{-1}$$

Noting that $\frac{T}{\theta}\frac{d\theta}{dz}=\Gamma_d-\Gamma$:

$$w\frac{T}{\theta_0}\frac{d\theta_0}{\partial z}=w\left(\Gamma_d-\Gamma\right)$$

What are typical values for Γ and Γ_d in the troposphere?

$$w\left(\Gamma_d-\Gamma\right)\sim W\left(\Gamma_d-\Gamma\right)\sim\left(10^{-2}\text{ m s}^{-1}\right)\left(4\text{ deg C km}^{-1}\right)\sim 4\text{ deg C day}^{-1}$$

Since $\frac{J}{c_pT}$ is less than the other terms in the thermodynamic energy equation:

$$\frac{1}{\theta_0}\left[\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+v\frac{\partial \theta'}{\partial y}\right]+\frac{w}{\theta_0}\frac{d\theta_0}{\partial z}\approx 0$$

$$\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+v\frac{\partial \theta'}{\partial y}+w\frac{d\theta_0}{\partial z}\approx 0$$

According to this equation what physical processes can cause local time variations of θ'?

What is the typical sign of $\frac{d\theta_0}{dz}$ in the atmosphere?

What does this imply about the impact of vertical motion on local changes in θ'?