Dynamic Meteorology - Introduction

Atmospheric dynamics – the study of atmospheric motions that are associated with weather and climate

We will consider the atmosphere to be a continuous fluid medium, or continuum.

Each "point" in the atmosphere will be made up of a large number of molecules, with certain properties.

These properties are assumed to be continuous functions of position and time.

The basic laws of thermodynamics and fluid mechanics can be expressed in terms of partial differential equations, with space and time as independent variables and the atmospheric properties as dependent variables.

Physical Dimensions and Units

Dimensional homogeneity – all terms in the equations that describe the atmosphere must have the same physical dimensions (units)

The four base units we will use are:

Property	Name	Symbol
Length	Meter (meter)	m
Mass	Kilogram	kg
Time	Second	s
Temperature	Kelvin	K

From these we will also use the following derived units:

Property	Name	Symbol
Frequency	Hertz	Hz (s^{-1})
Force	Newton	N (kg m s^{-2})
Pressure	Pascal	Pa (N m^{-2})
Energy	Joule	J (N m)
Power	Watt	W (J s^{-1})

Fundamental Forces

Newton's Second Law: F=ma

In atmospheric science it is typical to consider the force per unit mass acting on the atmosphere:

$$\frac{Force}{mass} = \vec{a}$$

In order to understand atmospheric motion (accelerations) we need to know what forces act on the atmosphere.

What are the fundamental forces of interest in atmospheric science?

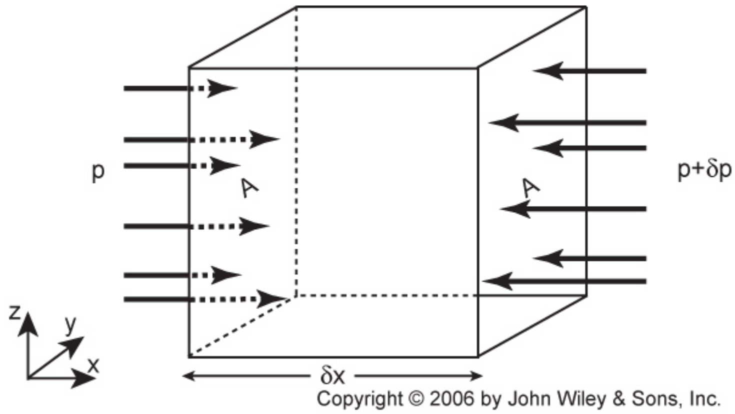
Body (or volume) force – a force that acts on the center of mass of a fluid parcel

Surface force – a force that acts across the boundary separating a fluid parcel from its surroundings. The magnitude of surface forces are independent of the mass of the parcel.

What are examples of body and surface forces?

Pressure Gradient Force

Example: Real-time weather map



The force exerted on the left face of this air parcel due to pressure is:

$$pA = p\delta y\delta z$$

The force exerted on the right face of this air parcel due to pressure is:

$$-(p+\delta p)\delta y\delta z = -\left(p + \frac{\partial p}{\partial x}\delta x\right)\delta y\delta z$$

The net force exerted by pressure on this air parcel is the sum of these forces and is equal to:

$$-\frac{\partial p}{\partial x}\delta x\delta y\delta z$$

By dividing by the mass of the air parcel ($\rho\delta x\delta y\delta z$) we get the force per unit mass due to changes in pressure (i.e. the pressure gradient force):

$$\vec{P}_g \bullet \vec{i} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

We can write all three components of the pressure gradient force as:

$$\vec{P}_g = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)$$

In vector form this can be expressed as:

$$\vec{P}_g = -\frac{1}{\rho}\nabla p$$

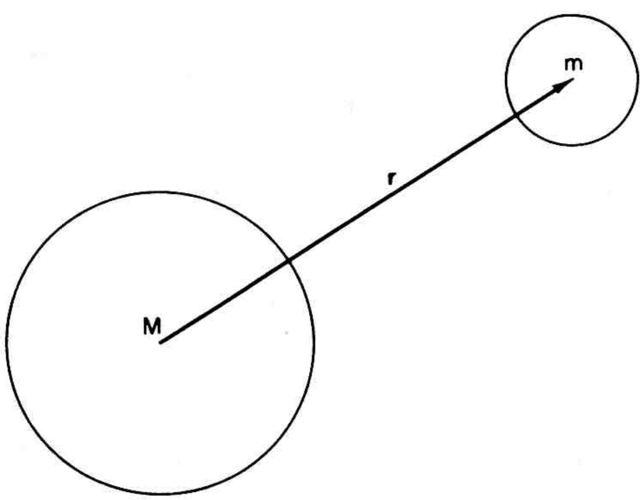
where $\nabla p = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}$

In what direction does this force act relative to locations with high and low pressure?

Example: Direction and magnitude of the pressure gradient force from a weather map

Gravitational Force

Newton's law of universal gravitation – any two elements of mass in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance separating them



$$\vec{F}_g = -\left(\frac{GMm}{r^2}\right)\left(\frac{\vec{r}}{r}\right)$$

G – gravitational constant (= 6.673×10^{-11} N m^2 kg^{-2})
M – mass of Earth (=5.988×10^{24} kg)
m – mass of air parcel
r – distance between objects

The gravitational force exerted on a unit mass of the atmosphere is:

$$\frac{\vec{F}_g}{m} \equiv \vec{g}^* = -\left(\frac{GM}{r^2}\right)\left(\frac{\vec{r}}{r}\right)$$

The distance *r* is given by:

$r = a + z$, where

a = mean radius of the earth (= 6.37×10^6 m)
z = distance above sea level

The gravitational force per unit mass of atmosphere at sea level is:

$\vec{g}_0^* = -\left(\frac{GM}{a^2}\right)\left(\frac{\vec{r}}{r}\right)$ and $\left|\vec{g}_0^*\right| = g_0^*$ = 9.85 m s^{-2}

At height z $\vec{g}^*$ is given by

$$\vec{g}^* = -\left(\frac{GM}{(a+z)^2}\right)\left(\frac{\vec{r}}{r}\right) = -\left(\frac{GM}{(a^2)\left(1+\frac{z}{a}\right)^2}\right)\left(\frac{\vec{r}}{r}\right) = \frac{\vec{g}_0^*}{\left(1+\frac{z}{a}\right)^2}$$

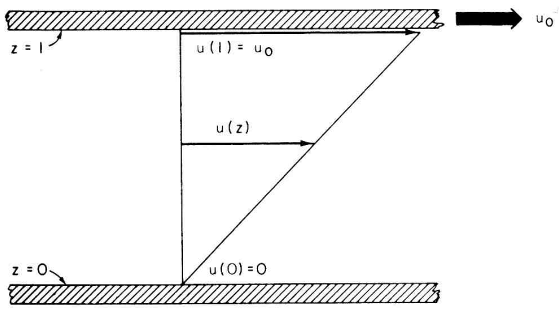
For meteorological applications $z \ll a$ and $\vec{g}^* \approx \vec{g}_0^*$

Therefore we can treat the gravitational force as a constant.

Viscous Force

Viscosity – internal friction which causes a fluid to resist the tendency to flow

Consider the fluid illustrated below, confined between two plates:



The fluid in contact with the plates moves at the speed of the plate.

The force required to keep the upper plate moving is:

$F = \frac{\mu A u_0}{l}$, where

μ = dynamic viscosity coefficient (a constant of proportionality)
A = area of the plate
u_0 = speed of upper plate
l = distance between the plates

In a steady state the force exerted on the plate is exactly equal to the force that the plate exerts on the fluid in contact with the plate and exactly opposite the force exerted by the fluid on the plate.

Each layer of fluid exerts the same force on the layer of fluid below.

The force can also be expressed as:

$F = \mu A \frac{\delta u}{\delta z}$, where

$\delta u = \frac{u_0}{l} \delta z$ and δz is the layer of fluid being considered.

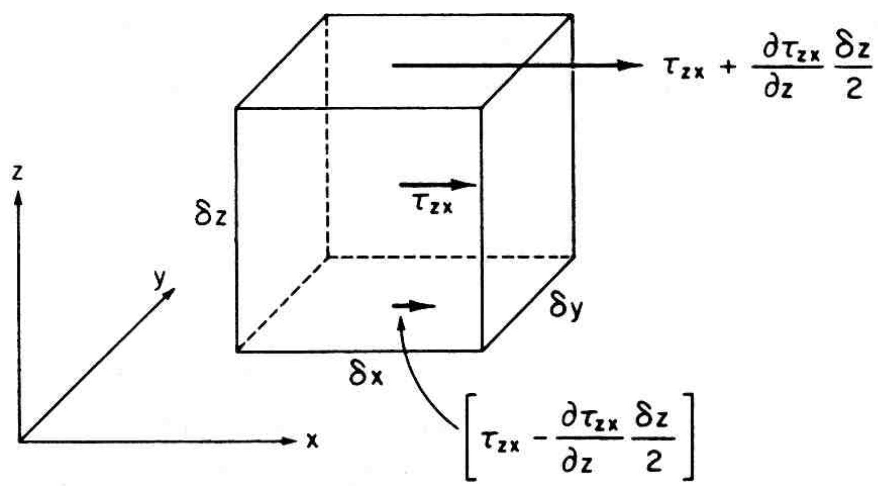
The shearing stress (τ_{zx}), defined as the viscous force per unit area is:

$$\tau_{zx} = \lim_{\delta z \to 0} \mu \frac{\delta u}{\delta z} = \mu \frac{\partial u}{\partial z}$$

The subscript *zx* indicates that this is the shearing stress in the *x* direction due to the vertical (*z*) shear of the *x* velocity component (*u*) of the flow.

What is the physical interpretation of the shearing stress?

Now, consider the more general case where τ_{zx} varies in the vertical:



The shear stress acting across the top boundary, on the fluid below, is:

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}$$

while the shear stress acting across the bottom boundary, on the fluid above, is:

$$-\left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]$$

and the viscous force at each boundary is equal to the shear stress multiplied by the surface area of the boundary ($\delta x \delta y$).

Then the net viscous force acting on this volume in the *x*-direction is:

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z$$

The viscous force per unit mass is:

$$\frac{\frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z}{\rho \delta x \delta y \delta z} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

Assuming that μ is constant gives:

$\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} = \nu \frac{\partial^2 u}{\partial z^2}$, where

$\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient (=1.46x10^{-5} m^2 s^{-1} for standard atmospheric conditions at sea level)

The total viscous force due to shear stresses in all three directions for all three components of the flow, and commonly referred to as the frictional force, is given by:

$$F_{rx} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$F_{ry} = \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$F_{rz} = \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Since ν is small in the atmosphere, this force is negligible in the atmosphere, except where the vertical shear is large.

Where in the atmosphere is the vertical shear large enough that this force non-negligible?

Noninertial Reference Frames and “Apparent” Forces

Newton’s first law states that a mass in uniform motion relative to a fixed frame of reference will remain in uniform motion in the absence of any net force acting on the mass.

This type of motion is referred to as inertial motion.

Inertial reference frame – a frame of reference fixed in space

Is a frame of reference which is fixed relative to the earth an inertial reference frame?

Noninertial reference frame – a frame of reference which is undergoing an acceleration

Newton’s laws of motion can only be applied in a noninertial reference frame if the acceleration of the reference frame is taken into account.

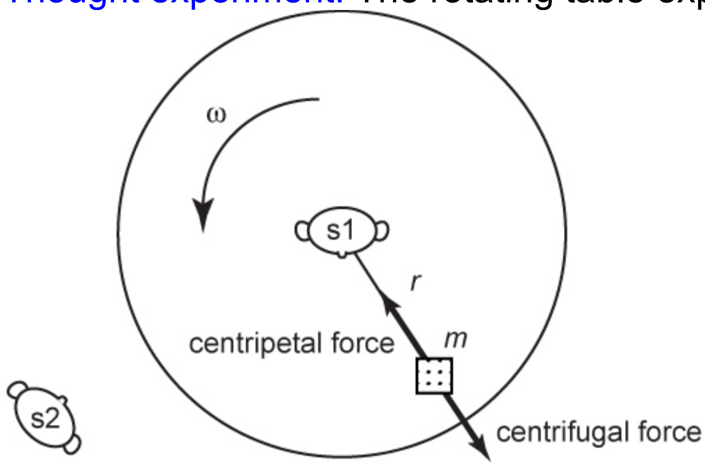
One way to do this is to introduce apparent forces that are due to the acceleration of the reference frame.

For a reference frame in uniform rotation the two apparent forces are:

- Centrifugal force
- Coriolis force

Centripetal and Centrifugal Forces

Thought experiment: The rotating table experiment



What happens to the object when it is placed on the rotating table?

In order to keep the object stationary on the rotating table observer s1 must exert a force equal to $mr\omega^2$ on a string attached to the object, where ω is the angular velocity of the table.

Centripetal force – inward radial force

From a fixed frame of reference (observer s2) this object is experiencing a centripetal acceleration.

From this frame of reference a single force (the centripetal force) acts on the object and causes a centripetal acceleration.

For a unit mass object this can be expressed as:

$$\frac{D\vec{V}}{Dt} = -\omega^2 r$$

From a frame of reference on the rotating table (observer s1) is the object experiencing an acceleration?

What forces are acting on the object in this frame of reference?

Centrifugal force – outward radial force

The centrifugal force arises only in observations taken in a rotating frame of reference and is due to the acceleration of the frame of reference.

For a unit mass object the centrifugal force is $\omega^2 r$

Gravity Force

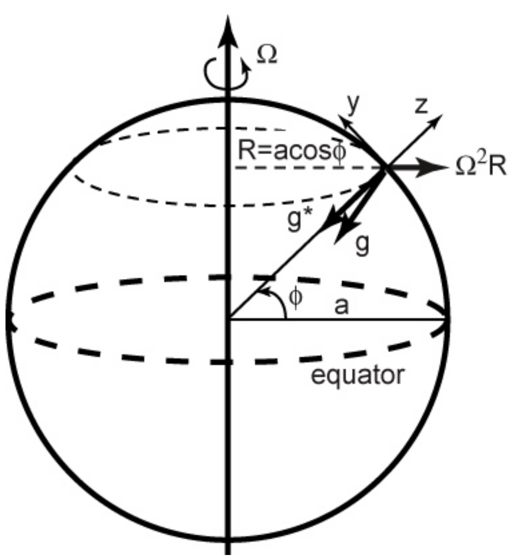
Angular velocity of rotation of the Earth:

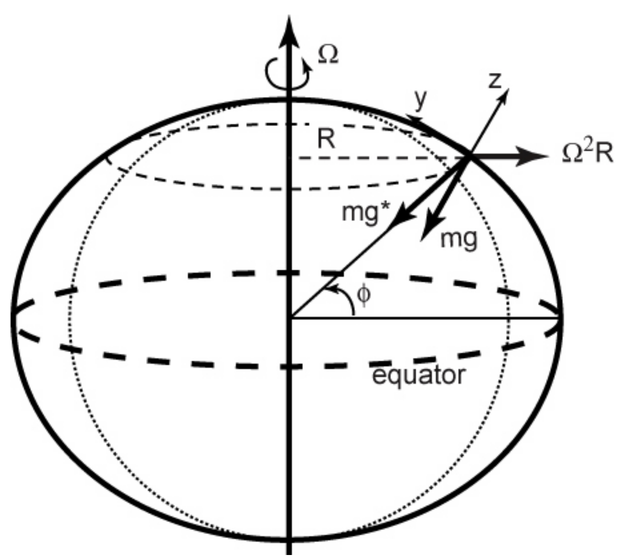
$$\Omega = \frac{2\pi \text{ rad}}{1 \text{ day}} = \frac{2\pi \text{ rad}}{86164 \text{ sec}} = 7.292 \times 10^{-5} \text{ rad s}^{-1}$$

Is an object at rest on the surface of the earth at rest in an inertial reference frame?

What is the direction and magnitude of the acceleration experienced by this object?

As was found for the rotating table experiment, to an observer on the earth (a noninertial reference frame) an object of unit mass experiences a centrifugal force equal to $\Omega^2 R$ directed radially outward from the axis of rotation.





This centrifugal force is a body force and can be combined with the gravity force to give an effective gravity, $\vec{g}$.

$$\vec{g} \equiv -g\vec{k} \equiv \vec{g}^* + \Omega^2 R$$

Effective gravity: g = 9.81 m s^{-2} at sea level

Gravity can be represented as the gradient of a potential function known as geopotential (Φ):

$$\vec{g} = -\nabla\Phi$$

Since $\vec{g} \equiv -g\vec{k}$ then $\Phi = \Phi(z)$ and $g = \dfrac{d\Phi}{dz}$ or $d\Phi = g dz$

The addition of the centrifugal force to the gravity force results in the effective gravity force not being directed towards the center of the Earth.

In reality the Earth is not a perfect sphere and the effective gravity is exactly normal to the surface of the Earth at all locations (neglecting topography). Therefore, the effective gravity force acts only in the *z*-direction.

Further, from the definition of geopotential, the gravity force is exactly normal to geopotential surfaces, and the surface of the earth, neglecting topography, is a geopotential surface.

This discussion has only considered an object at rest in a rotating frame of reference.

What happens when an object is in motion relative to a rotating frame of reference?

The Coriolis Force and the Curvature Effect

For an object in motion relative to a rotating frame of reference we need to consider:

- changes in the relative angular momentum of the object

- an additional centrifugal force

Conservation of angular momentum

For a rotating object of unit mass the angular momentum of the object is given by:

$$\tfrac{1}{2}R^2\omega$$

and will remain constant in the absence of a torque (force) acting on the object.

For an air parcel the angular velocity ω will be given by $\Omega + \dfrac{u}{R}$, where:

Ω is the angular velocity associated with the rotation of the earth and

$\dfrac{u}{R}$ is the angular velocity associated with zonal motion relative to the earth.

Then the angular momentum of this air parcel is given by $\tfrac{1}{2}R^2\left(\Omega + \dfrac{u}{R}\right)$

If this air parcel is displaced such that R changes by δR then the angular momentum will be given by:

$$\tfrac{1}{2}(R+\delta R)^2\left(\Omega + \frac{u+\delta u}{R+\delta R}\right)$$

and must be equal to the initial angular momentum.

This equality gives:

$$\tfrac{1}{2}R^2\left(\Omega+\frac{u}{R}\right)=\tfrac{1}{2}(R+\delta R)^2\left(\Omega+\frac{u+\delta u}{R+\delta R}\right)$$

What will the sign of δR be for an air parcel moving towards the North Pole (a meridional displacement)?

What does this imply about the sign of δu?

In solving for δu in this equation we will neglect all squared δ terms.

$$\tfrac{1}{2}R^2\left(\Omega+\frac{u}{R}\right)=\tfrac{1}{2}(R+\delta R)^2\left(\Omega+\frac{u+\delta u}{R+\delta R}\right)$$

$$R^2\Omega+R^2\frac{u}{R}=\left(R^2+2R\delta R+(\delta R)^2\right)\left(\Omega+\frac{u}{R+\delta R}+\frac{\delta u}{R+\delta R}\right)$$

$$R^2\Omega+R^2\frac{u}{R}=R^2\Omega+\frac{R^2u}{R+\delta R}+\frac{R^2\delta u}{R+\delta R}+2R\Omega\delta R+\frac{2R\delta Ru}{R+\delta R}+\frac{2R\delta R\delta u}{R+\delta R}$$

$$R^2\frac{u}{R}=\frac{R^2u}{R+\delta R}+\frac{R^2\delta u}{R+\delta R}+2R\Omega\delta R+\frac{2R\delta Ru}{R+\delta R}$$

Multiplying by $R+\delta R$ gives

$$R^2\frac{u}{R}(R+\delta R)=R^2u+R^2\delta u+2R\Omega\delta R(R+\delta R)+2R\delta Ru$$

$$R^2u+Ru\delta R=R^2u+R^2\delta u+2R^2\Omega\delta R+2R\Omega(\delta R)^2+2Ru\delta R$$

$$R^2\delta u=-2R^2\Omega\delta R-Ru\delta R$$

$$\delta u=-2\Omega\delta R-\frac{u\delta R}{R}$$

Noting that $R = a\cos\phi$ and $\delta R = -\delta y \sin\phi$ for a meridional displacement we get:

$$\delta u = 2\Omega \delta y \sin\phi + \frac{u\delta y \sin\phi}{a\cos\phi}$$

$$\delta u = 2\Omega \delta y \sin\phi + \frac{u}{a}\delta y \tan\phi$$

Dividing by δt gives

$$\frac{\delta u}{\delta t} = 2\Omega \frac{\delta y}{\delta t}\sin\phi + \frac{u}{a}\frac{\delta y}{\delta t}\tan\phi$$

Taking the limit as $\delta t \to 0$ we then have:

$$\frac{Du}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a}\tan\phi,$$

where we have used $v = \frac{Dy}{Dt}$.

This equation indicates that conservation of angular momentum results in a zonal acceleration (Du/Dt) for objects moving in a meridional (north/south) direction.

For a vertical displacement of an air parcel $\delta R = \delta z \cos\phi$ and

$$\delta u = -2\Omega \delta R - \frac{u\delta R}{R}$$

$$\delta u = -2\Omega \delta z \cos\phi - \frac{u\delta z \cos\phi}{R}$$

Again, dividing by δt and taking the limit as $\delta t \to 0$ we get

$$\frac{Du}{Dt} = -2\Omega w \cos\phi - \frac{uw}{a},$$ where we have used $w = \frac{Dz}{Dt}$.

This equation indicates that conservation of angular momentum also results in a zonal acceleration for objects moving in the vertical direction.

Conservation of angular momentum does not result in an acceleration for objects moving in a purely zonal direction. Why?

These apparent forces that arise due to conservation of angular momentum are part of the Coriolis force.

What is the centrifugal force acting on an object of unit mass moving towards the east?

The angular velocity due to the zonal (u) motion is given by $\frac{u}{R}$ and the total centrifugal force acting on this mass is:

$$CF_{total} = \left(\Omega + \frac{u}{R}\right)^2 \vec{R}$$

$$= \Omega^2 \vec{R} + \frac{2\Omega u}{R}\vec{R} + \frac{u^2}{R^2}\vec{R}$$

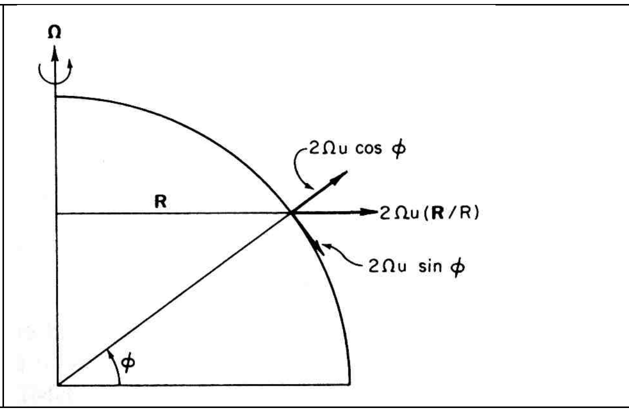
The first term on the RHS of this equation is the same as the centrifugal force for an object at rest, and is incorporated into the effective gravity.

The remaining terms on the RHS of this equation can be partitioned into meridional and vertical components that lead to an acceleration of the v and w components of the wind.

For the second term on the RHS of this equation, this partitioning is:

Meridional: $\frac{Dv}{Dt} = -2\Omega u \sin\phi$

Vertical: $\frac{Dw}{Dt} = 2\Omega u \cos\phi$



These terms represent the Coriolis force for zonal motion.

Similarly the last term, which represents curvature effects, can be partitioned into:

Meridional: $\frac{Dv}{Dt} = -\frac{u^2}{R}\sin\phi = -\frac{u^2 \sin\phi}{a\cos\phi} = -\frac{u^2}{a}\tan\phi$

Vertical: $\frac{Dw}{Dt} = \frac{u^2}{R}\cos\phi = \frac{u^2\cos\phi}{a\cos\phi} = \frac{u^2}{a}$

These terms arise due to the curvature of the earth.

The Coriolis force terms from conservation of angular momentum and the centrifugal force due to the motion of an air parcel can be combined to give:

$$\frac{F_{Coriolis}}{m} = (2\Omega v\sin\phi - 2\Omega w\cos\phi)\vec{i} - 2\Omega u\sin\phi\vec{j} + 2\Omega u\cos\phi\vec{k}$$

Using vector notation, and noting that $\vec{\Omega} = \Omega\cos\phi\vec{j} + \Omega\sin\phi\vec{k}$ the Coriolis force can also be expressed as:

$$\frac{F_{Coriolis}}{m} = -2\vec{\Omega}\times\vec{u}$$

Considering the components of the Coriolis force that arise from the horizontal wind (*u* and *v*) and that give rise to horizontal accelerations $\left(\frac{Du}{Dt}, \frac{Dv}{Dt}\right)$ we have:

$$\left(\frac{Du}{Dt}\right)_{Co} = 2\Omega v\sin\phi = fv$$
$$\left(\frac{Dv}{Dt}\right)_{Co} = -2\Omega u\sin\phi = -fu,$$

where $f = 2\Omega\sin\phi$ is the Coriolis parameter.

In what direction is a zonal wind accelerated by the Coriolis force?

In what direction is a meridional wind accelerated by the Coriolis force? What is the direction of acceleration due to the Coriolis force relative to the wind direction?

The *x* and *y* components of the Coriolis force can also be written as:

$$\left(\frac{D\vec{V}}{Dt}\right)_{Co} = -f\vec{k} \times \vec{V},$$

where $\vec{V} \equiv (u,v) = u\vec{i} + v\vec{j}$

What is the direction of $-f\vec{k} \times \vec{V}$?

Physical interpretation of the Coriolis force: Why does conservation of angular momentum and the centrifugal force lead to this direction of the Coriolis force relative to the wind?

The Coriolis force is negligible for motions with time scales that are very short compared to the period of the Earth's rotation.

Example: Calculate the deflection of a dart thrown due east at a latitude of 40°N at a speed of 30 m s^{-1}. The distance to the dart board is 2.4 m.

Example: The Coriolis force on the roof of Duane Physics

Note: The horizontal components of the wind can be calculated from:

$$u = -|\vec{V}|\sin(WD)$$
$$v = -|\vec{V}|\cos(WD)$$

Constant Angular Momentum Oscillations

Consider an air parcel initially propelled in the positive *x* direction with a speed of *V* that is acted upon only by the Coriolis force.

The equations that describe this motion are:

$$\left(\frac{Du}{Dt}\right)_{Co} = fv$$

$$\left(\frac{Dv}{Dt}\right)_{Co} = -fu$$

Example: Use these equations to calculate the position (*x*,*y*) of this air parcel as a function of time.

Start by combining these equations into a single differential equation.

Using $v = \frac{1}{f}\frac{Du}{Dt}$ substitute into $\frac{Dv}{Dt} = -fu$ to get

$$\frac{D\left(\frac{1}{f}\frac{Du}{Dt}\right)}{Dt} = -fu$$

$$\frac{1}{f}\frac{D^2u}{Dt^2} + fu = 0$$

$$\frac{D^2u}{Dt^2} + f^2u = 0$$

The solution for this differential equation is:

$$u = V\cos(ft)$$

$$v = -V\sin(ft)$$

This solution can now be substituted into the original equations to give:

$$\frac{Du}{Dt} = \frac{D\left[V\cos(ft)\right]}{Dt} = fv \qquad \frac{Dv}{Dt} = \frac{D\left[-V\sin(ft)\right]}{Dt} = -fu$$

$$-fV\sin(ft) = fv \qquad -fV\cos(ft) = -fu$$

$$-V\sin(ft) = \frac{Dy}{Dt} \qquad V\cos(ft) = \frac{Dx}{Dt}$$

where we have used $v = \frac{Dy}{Dt}$ and $u = \frac{Dx}{Dt}$

These equations can now be integrated in time to give the position (*x,y*) as a function of time:

$$\int_{y_0}^{y} Dy = \int_0^t -V\sin(ft)Dt \qquad \int_{x_0}^{x} Dx = \int_0^t V\cos(ft)Dt$$

$$y - y_0 = \frac{V}{f}\cos(ft)\Big|_0^t \qquad x - x_0 = \frac{V}{f}\sin(ft)\Big|_0^t$$

$$y - y_0 = \frac{V}{f}\cos(ft) - \frac{V}{f} \qquad x - x_0 = \frac{V}{f}\sin(ft)$$

$$y = y_0 + \frac{V}{f}\cos(ft) - \frac{V}{f} \qquad x = x_0 + \frac{V}{f}\sin(ft)$$

These equations describe a circle with:

Center at $\left(x_0, y_0 - \frac{V}{f}\right)$

and radius $R = \frac{V}{f}$

How long does it take for this air parcel to complete one revolution around this circular path?

Pressure as a Vertical Coordinate

From the hydrostatic equation:

$$dp = -\rho g dz$$

we see that there is a monotonic relationship between pressure and height in any column of the atmosphere (i.e. there is a unique value of pressure at each height in the atmosphere, and that the pressure decreases with increasing height).

Therefore we can use pressure as our vertical coordinate rather than height.

Similarly, since $d\Phi = -\alpha dp = -\frac{RTdp}{p}$ we could also use pressure as a vertical coordinate rather than geopotential height.

With pressure as the vertical coordinate the properties of the atmosphere will then depend on horizontal position (*x*,*y*), pressure *p*, and time *t*:

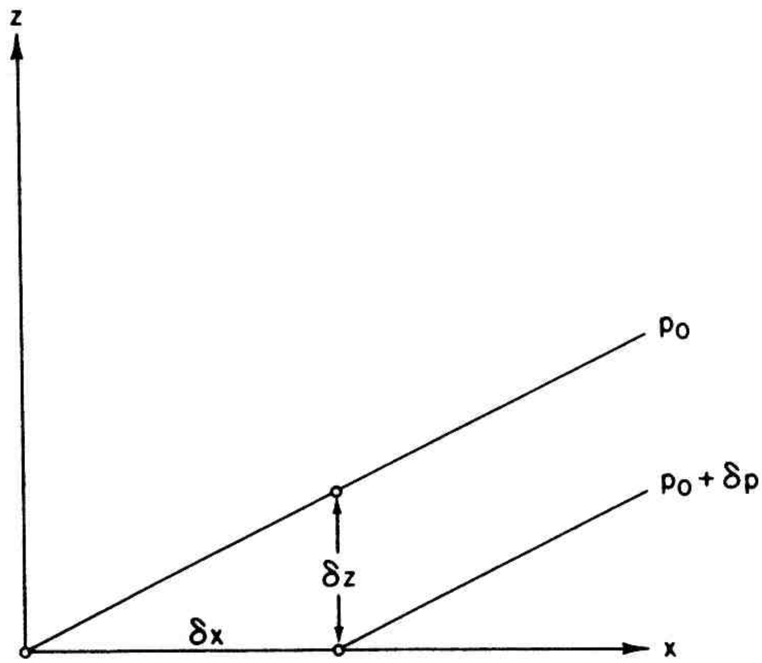
T(x,y,p,t), *u(x,y,p,t), v(x,y,p,t), Φ(x,y,p,t)*, etc.

Consider the horizontal pressure gradient force in the *x*-direction, given by:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

When height (*z*) is used as a vertical coordinate this horizontal derivative is evaluated by holding *z* constant.

When we use pressure as a vertical coordinate we now need to evaluate this derivative along a constant pressure surface.



From the geometry in this diagram we note:

$$\left[\frac{(p_0+\delta p)-p_0}{\delta x}\right]_z = \left[\frac{p_0-(p_0+\delta p)}{\delta z}\right]_x \left(\frac{\delta z}{\delta x}\right)_p$$

In the limit as $\delta x, \delta z \rightarrow 0$ we get

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

The hydrostatic equation can be used to replace $-\left(\frac{\partial p}{\partial z}\right)_x$ to give

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho g \left(\frac{\partial z}{\partial x}\right)_p$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$

This equation indicates that the horizontal pressure gradient (at constant height) is equivalent to the horizontal geopotential gradient at constant pressure.

Similarly $-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p$

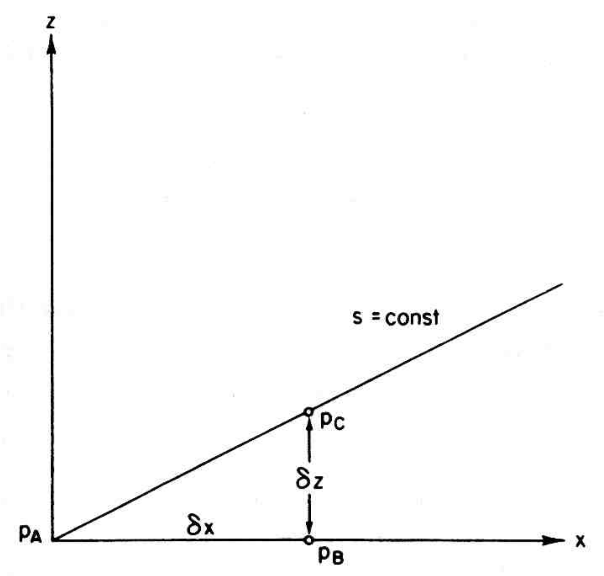
Note that in constant pressure coordinates (isobaric coordinates) the pressure gradient force term no longer includes density.

Example: Compare the pressure gradient force calculated from a constant pressure and a constant height map

A Generalized Vertical Coordinate

We can use any single-valued monotonic function of pressure or height as a vertical coordinate.

Numerical weather prediction models often use $\sigma \equiv p(x,y,z,t)/p_s(x,y,t)$ as a vertical coordinate since $\sigma = 1$ by definition at the surface of the earth and thus this coordinate is terrain following.



From this figure:

$$\left(\frac{p_c - p_a}{\delta x}\right)_s = \left(\frac{p_c - p_b}{\delta z}\right)_x \left(\frac{\delta z}{\delta x}\right)_s + \left(\frac{p_b - p_a}{\delta x}\right)_z$$

In the limit as $\delta x, \delta z \rightarrow 0$ we get

$$\left(\frac{\partial p}{\partial x}\right)_s = \left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z$$

Noting that $\frac{\partial p}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial p}{\partial s}$ this can be rewritten as

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial s}{\partial z}\frac{\partial p}{\partial s}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z$$

This equation can be used to transform the atmospheric governing equations expressed using *z* as a vertical coordinate to any other vertical coordinate.

Example: Use this equation to show that $-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$ when converting from a height vertical coordinate to a pressure vertical coordinate